Constructor University
Fall 2025

December 17, 2025

Elements of Linear Algebra
Final Exam

Instructions:

- The exam has 16 multiple choice questions (several answers can be correct!) and 2 longer questions. The total number of points is 116.
- For the multiple choice questions, it is sufficient to mark the final answer(s) only. (No solution steps necessary.) There are no negative points, but of course there are fewer points if wrong answers are selected, or if right answers are not selected.
- For the longer exercises 17 and 18, you need to show your work, i.e., carefully write down the steps of your solution. You will receive points not just based on your final answer, but on the correct steps in your solution.
- No tools or other resources are allowed for this exam. In particular, no notes and no calculators.
- You are free to refer to any results proven in class or the homework sheets unless stated otherwise (and unless the problem is to reproduce a result from class or the homework sheets).

Code of Academic Integrity

I pledge that the answers of this exam are my own work without the assistance of others or the usage of unauthorized material or information.

Sign to confirm that you adhere to the Academic Integrity Code:

Name: ____

Signature: ____________________________________

Matric./Student No.: __________________________

1. **(4 points)** A matrix has characteristic polynomial $p(x) = x^2 - 2x + 5$. Find the roots of $p(x)$, i.e., the solutions to $p(x) = 0$.
   A. * The roots are $x_1 = 1 + 2i$ and $x_2 = 1 - 2i$.
   B. The roots are $x_1 = 0$ and $x_2 = 2$.
   C. The roots are $x_1 = -1$ and $x_2 = 1$.
   D. The roots are $x_1 = 2 + i$ and $x_2 = 2 - i$.
   E. There is only one root $x_1 = 1$.
   F. The roots are $x_1 = -1 + 2i$ and $x_2 = 1 + 2i$.

2. **(6 points)** Consider the vectors

$$a = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

   Compute their scalar (dot) product, their cross product, and the lengths of the vectors.
   A. * The cross product is $a \times b = \begin{pmatrix} 0 \\ 10 \\ -5 \end{pmatrix}$.
   B. The scalar (dot) product is $a \cdot b = 11$.
   C. * The length of a is $|a| = \sqrt{21}$ and the length of b is $|b| = \sqrt{14}$.
   D. The cross product is $a \times b = \begin{pmatrix} 8 \\ 14 \\ 7 \end{pmatrix}$.
   E. The length of a is $|a| = \sqrt{7}$ and the length of b is $|b| = \sqrt{6}$.
   F. * The scalar (dot) product is $a \cdot b = 13$.

3. **(6 points)** Which of the following statements are true?

A. The vectors $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\2\\2\end{pmatrix}$ are linearly independent.

B. The vectors $\begin{pmatrix}0\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ are a basis of $\mathbb{R}^3$.

C. The vectors $\begin{pmatrix}1\\2\\2\end{pmatrix}, \begin{pmatrix}-1\\-2\\-2\end{pmatrix}$ are linearly independent.

D. * The vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ are a basis of $\mathbb{R}^3$.

E. * The vectors $\begin{pmatrix}2\\2\\3\end{pmatrix}, \begin{pmatrix}1\\3\\2\end{pmatrix}$ are linearly independent.

F. * The vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\2\\2\end{pmatrix}$ are a basis of $\mathbb{R}^3$.

4. **(4 points)** Recall that A^T denotes the transpose of the matrix A. Now let

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$$

Compute $(Av)^T$.

A. $(Av)^T = 10$.

B. $(Av)^T = \begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix}$.

C. $(Av)^T = \begin{pmatrix} 10 & 8 & 8 \end{pmatrix}$.

D. $(Av)^T = \begin{pmatrix} 10 \\ 8 \\ 8 \end{pmatrix}$.

E. $(Av)^T = \begin{pmatrix} 8 \\ 10 \end{pmatrix}$.

F. * $(Av)^T = \begin{pmatrix} 10 & 8 \end{pmatrix}$.

5. **(4 points)** Let

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 3 & 4 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Calculate the matrix product AB.

A. * $\begin{pmatrix} 0 & 2 \\ 2 & 5 \end{pmatrix}$.

B. $\begin{pmatrix} 2 & 5 \\ 2 & 0 \end{pmatrix}$.

C. $\begin{pmatrix} 2 & 2 & 5 \\ 2 & 5 & 2 \\ 1 & -1 & 1 \end{pmatrix}$.

D. $\begin{pmatrix} 5 & 2 \\ 2 & 0 \end{pmatrix}$.

E. $\begin{pmatrix} 2 & 2 & 5 \\ 4 & 5 & 2 \end{pmatrix}$.

F. $\begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$.

6. **(4 points)** Which of the following describes the solution(s) to the system of linear equations

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 4, \\ x_2 + 2x_3 &= 3, \\ -x_2 - 2x_3 &= 0. \end{aligned}$$

A. The unique solution is $x = (2, 1, 1)$.

B. * The system of equations has no solutions.

C. The unique solution is $x = (0, 1, 1)$.

D. The system of equations has infinitely many solutions $x = (1, 0, 0) + \lambda(1, 1, 1)$.

E. The system of equations has infinitely many solutions $x = (2, 1, 0) + \lambda(1, 2, 1)$.

F. The unique solution is $x = (1, 1, 1)$.

7. **(6 points)** A system of linear equations $Ax = b$ has been brought, through Gaussian elimination, into the reduced row-echelon form (in augmented matrix notation)

$$\left(\begin{array}{cccc|c} 1 & 7 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 1 & 0 & 3 \\ 0 & 4 & 0 & 1 & 1 \end{array}\right).$$

Which of the following statements are true?

A. The rank of A is 2.

B. The general solution to this system can be written as $x = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 1 \\ 6 \\ 4 \end{pmatrix}$, for $\lambda \in \mathbb{R}$.

C. The rank of A is 1.

D. The general solution to this system can be written as $x = \begin{pmatrix} 7 \\ -1 \\ 6 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 3 \\ 1 \end{pmatrix}$, for $\lambda \in \mathbb{R}$.

E. * The general solution to this system can be written as $x = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -1 \\ 6 \\ 4 \end{pmatrix}$, for $\lambda \in \mathbb{R}$.

F. * The rank of A is 3.

8. **(6 points)** Let A be an $n \times n$ matrix. Which of the following statements are equivalent to "The matrix A is invertible"?

A. * The rows of A are linearly independent.

B. * The columns of A are linearly independent.

C. The system of linear equations $Ax = 0$ has infinitely many solutions.

D. The determinant of A is 0.

E. A has at least one eigenvalue zero.

F. * The system of linear equations $Ax = 0$ has the unique solution $x = 0$.

9. **(4 points)** Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

A. The matrix A is not invertible.

B. $A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -3 & 1 \\ 1 & 2 & -1 \\ -1 & 1 & 2 \end{pmatrix}$.

C. $A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -3 & 1 \\ 1 & 2 & -1 \\ -1 & 2 & 1 \end{pmatrix}$.

D. $A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -3 & 1 \\ -1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$.

E. * $A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -3 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$.

F. $A^{-1} = \begin{pmatrix} 1 & -3 & 2 \\ 2 & 1 & -1 \\ -1 & 1 & 3 \end{pmatrix}$.

10. **(4 points)** Compute the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}.$$

A. The eigenvalues are $\lambda_1 = 1 - i$ and $\lambda_2 = 1 + i$.

B. The eigenvalues are $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$.

C. The eigenvalues are $\lambda_1 = -2i$ and $\lambda_2 = 2i$.

D. The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 1$.

E. * The eigenvalues are $\lambda_1 = -i$ and $\lambda_2 = i$.

F. The eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 2$.

11. **(6 points)** Let A be an $n \times n$ matrix. Which of the following statements are true?

A. * The determinant of A is given by the product of all eigenvalues, including their multiplicities.

B. A has exactly n distinct eigenvalues.

C. The determinant of A is given by the sum of all eigenvalues, including their multiplicities.

D. A is diagonalizable if and only if $\det A = 0$.

E. * A is diagonalizable if and only if the algebraic multiplicity equals the geometric multiplicity for every eigenvalue.

F. * If $\lambda \neq 0$ is an eigenvalue of A, and A is invertible, then λ^{-1} is an eigenvalue of A^{-1}.

12. **(6 points)** Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 4 & 2 \\ 2 & 4 & 2 & 2 \\ 4 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix}.$$

Which of the following is true?

A. A is unitary.

B. * A is normal.

C. * A is real symmetric.

D. A is skew-Hermitian.

E. A is orthogonal.

F. * A is Hermitian.

13. **(6 points)** Which of the following statements are equivalent to "U is a unitary $n \times n$ matrix"?

A. * $|Ux| = |x|$ for all vectors $x \in \mathbb{C}^n$.

B. * All columns of U are orthonormal.

C. All eigenvalues of U are purely imaginary.

D. $U^* = U$.

E. * $U^* = U^{-1}$.

F. $U^*U = UU^*$.

14. **(4 points)** Compute the LU decomposition of the matrix

$$A = \begin{pmatrix} -1 & 1 \\ -2 & 5 \end{pmatrix}$$

such that all diagonal entries of L are one. What are the diagonal entries of U?

A. * U has diagonal entries $-1, 3$.
B. U has diagonal entries $-1, 2$.
C. U has diagonal entries $1, 3$.
D. U has diagonal entries $-1, 1$.
E. U has diagonal entries $1, 1$.
F. U has diagonal entries $1, 2$.

15. **(4 points)** Consider the matrix

$$A = \begin{pmatrix} 3 & 6 \\ 3 & -2 \end{pmatrix}.$$

Which of the following is a valid QR-decomposition?

A. $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad R = \sqrt{2} \begin{pmatrix} 3 & 14 \\ 0 & -8 \end{pmatrix}.$
B. $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad R = \sqrt{2} \begin{pmatrix} -\frac{3}{5} & 2 \\ \frac{9}{5} & 2 \end{pmatrix}.$
C. $Q = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 3 & 2 \\ 0 & 4 \end{pmatrix}.$
D. * $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad R = \sqrt{2} \begin{pmatrix} 3 & 2 \\ 0 & 4 \end{pmatrix}.$
E. $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad R = \sqrt{2} \begin{pmatrix} 3 & 0 \\ 2 & 4 \end{pmatrix}.$
F. None of the options are valid QR decompositions.

16. **(6 points)** Consider the matrix

$$A = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 17/10 & 1/10 & -17/10 & -1/10 \\ 3/5 & 9/5 & -3/5 & -9/5 \end{pmatrix}.$$

A has a singular value decomposition $A = U\Sigma V^*$ with

$$U = \frac{1}{5}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \quad V^* = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Which of the following statements are true?

A. * $\text{rank}(A) = 3$.

B. * The singular values are 4, 3, and 2.

C. $\text{rank}(A) = 1$.

D. * U is unitary.

E. $\text{rank}(A) = 2$.

F. V is Hermitian.

G. The singular values are 5, 4, 3, and -4.

17. **(18 points)**

Use Gaussian elimination to find the general solution to the system of linear equations

$$\begin{aligned} x_1 - x_2 + 3x_4 &= 2, \\ x_2 - x_4 &= 2, \\ 2x_1 + x_3 + 7x_4 &= 9, \\ 3x_1 + x_3 + 9x_4 &= 13. \end{aligned}$$

(Here, you need to write down all steps of your solution in order to receive full points.)

Solution. We use the augmented matrix notation and bring the system into reduced row-echelon form. We find

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 3 & 2 \\ 0 & 1 & 0 & -1 & 2 \\ 2 & 0 & 1 & 7 & 9 \\ 3 & 0 & 1 & 9 & 13 \end{array}\right)$$

Take $-2R_1 + R_3 \to R_3$ and $-3R_1 + R_4 \to R_4$:

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 3 & 2 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 2 & 1 & 1 & 5 \\ 0 & 3 & 1 & 0 & 7 \end{array}\right)$$

Take $-2R_2 + R_3 \to R_3$ and $-3R_2 + R_4 \to R_4$:

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 3 & 2 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 & 1 \end{array}\right)$$

Take $-R_3 + R_4 \to R_4$ and $R_2 + R_1 \to R_1$:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The general solution to this system can be written as

$$x = \begin{pmatrix} 4 \\ 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \\ -1 \end{pmatrix},$$

for $\lambda \in \mathbb{R}$.

18. **(18 points)**

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}.$$

Diagonalize the matrix A, and additionally compute all singular values of A. (Here, you need to write down all steps of your solution in order to receive full points.)

Solution. We first compute the eigenvalues. We find

$$0 = \det(A - \lambda 1) = \det \begin{pmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{pmatrix} = (1-\lambda)(-\lambda) - 2 = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2).$$

Hence, the eigenvalues are $\lambda_- = -1$ and $\lambda_+ = 2$. The corresponding eigenvectors are the following. For $\lambda_- = -1$, we need to solve

$$\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 1 & 1 & 0 \end{array}\right).$$

For example, let us choose $v_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. For $\lambda_+ = 2$, we need to solve

$$\left(\begin{array}{cc|c} -1 & 2 & 0 \\ 1 & -2 & 0 \end{array}\right).$$

For example, let us choose $v_+ = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

The diagonalizing matrix V has the eigenvectors as columns, i.e.,

$$V = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}.$$

Its inverse is

$$V^{-1} = \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}.$$

Hence, a diagonalization is

$$A = VDV^{-1} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}.$$

To compute the singular values, we compute

$$A^*A = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}.$$

Next, we compute the eigenvalues of A^*A. We find

$$0 = \det(A^*A - \lambda 1) = \det \begin{pmatrix} 2-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} = (2-\lambda)(4-\lambda) - 4 = \lambda^2 - 6\lambda + 4.$$

Hence, the eigenvalues are $\lambda_\pm = 3 \pm \sqrt{5}$. The singular values $\sigma_\pm$ are the square roots of these eigenvalues, hence they are $\sigma_\pm = \sqrt{3 \pm \sqrt{5}}$.