

Elements of Linear Algebra

Final Exam (Make-up)

Instructions:

- The exam has 16 multiple choice questions (several answers can be correct!) and 2 longer questions. The total number of points is 118.
- For the multiple choice questions, it is sufficient to mark the final answer(s) only. (No solution steps necessary.) There are no negative points, but of course there are fewer points if wrong answers are selected, or if right answers are not selected.
- For the longer exercises 17 and 18, you need to show your work, i.e., carefully write down the steps of your solution. You will receive points not just based on your final answer, but on the correct steps in your solution.
- No tools or other resources are allowed for this exam. In particular, no notes and no calculators.
- You are free to refer to any results proven in class or the homework sheets unless stated otherwise (and unless the problem is to reproduce a result from class or the homework sheets).

Code of Academic Integrity

I pledge that the answers of this exam are my own work without the assistance of others or the usage of unauthorized material or information.

Sign to confirm that you adhere to the Academic Integrity Code:

Name: _____

Signature: _____

Matric./Student No.: _____

1. **(4 points)** A matrix has characteristic polynomial $p(x) = x^2 + 4x + 5$. Find the roots of $p(x)$, i.e., the solutions to $p(x) = 0$.
- A. * The roots are $x_1 = -2 - i$ and $x_2 = -2 + i$.
 - B. The roots are $x_1 = 3 + \sqrt{2}i$ and $x_2 = 3 - \sqrt{2}i$.
 - C. There is only one root $x_1 = 2$.
 - D. The roots are $x_1 = 1$ and $x_2 = -5$.
 - E. The roots are $x_1 = -3$ and $x_2 = -1$.
 - F. The roots are $x_1 = 2 + i$ and $x_2 = -2 + i$.

2. **(6 points)** Consider the vectors

$$a = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}.$$

Compute their scalar (dot) product, their cross product, and the lengths of the vectors.

- A. * The cross product is $a \times b = \begin{pmatrix} 8 \\ 9 \\ -5 \end{pmatrix}$.
- B. The length of a is $|a| = \sqrt{5}$ and the length of b is $|b| = \sqrt{7}$.
- C. The scalar (dot) product is $a \cdot b = 0$.
- D. The cross product is $a \times b = \begin{pmatrix} 5 \\ 9 \\ -8 \end{pmatrix}$.
- E. * The scalar (dot) product is $a \cdot b = 1$.
- F. * The length of a is $|a| = 3$ and the length of b is $|b| = \sqrt{19}$.

3. (6 points) Which of the following statements are true?

A. The vectors $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are a basis of $\mathbb{R}^3$.

B. * The vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ are linearly independent.

C. * The vectors $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$ are a basis of $\mathbb{R}^3$.

D. * The vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ are linearly independent.

E. The vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 4 \\ 8 \\ 12 \end{pmatrix}$ are a basis of $\mathbb{R}^3$.

F. The vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$ are a basis of $\mathbb{R}^3$.

4. (4 points) Consider the point $y = (5, 2, 4)$ and the line

$$x = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Which of the following describes the plane that contains that line and the point y ?

- A. $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$
- B. $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$
- C. $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}.$
- D. $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}.$
- E. $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$
- F. $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}.$

5. (4 points) Let

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}.$$

Calculate the matrix product AB^T (where B^T denotes the transpose of B).

A. The matrix product cannot be computed because the dimensions are not right.

B. $\begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}.$

C. $\begin{pmatrix} 4 & 2 & 3 \\ 3 & 4 & -3 \end{pmatrix}.$

D. $\begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix}.$

E. $\begin{pmatrix} -1 & 4 \\ 4 & -1 \end{pmatrix}.$

F. $\begin{pmatrix} 9 & 13 \\ 9 & 4 \end{pmatrix}.$

6. (6 points) [Taken out of grading due to typo.] Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 &= 1, \\ 3x_1 + \alpha x_2 &= 3, \end{aligned}$$

with parameter $\alpha \in \mathbb{R}$. Which of the following statements are true?

A. For $\alpha = 1$, the system of equations has a unique solution.

B. For $\alpha = 1$, the system of equations has no solutions.

C. * For $\alpha = 1$, the system of equations has infinitely many solutions.

D. For $\alpha \neq 1$, the system of equations has infinitely many solutions.

E. For $\alpha \neq 1$, the system of equations has no solutions.

F. * For $\alpha \neq 1$, the system of equations has a unique solution.

7. **(6 points)** A system of linear equations $Ax = b$ has been brought, through Gaussian elimination, into the reduced row-echelon form (in augmented matrix notation)

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 5 \\ 0 & 1 & 1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 8 \end{array} \right).$$

Which of the following statements are true?

A. The general solution to this system can be written as $x = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 6 \\ 0 \\ 8 \end{pmatrix}$, for $\lambda \in \mathbb{R}$.

B. The general solution to this system can be written as $x = \begin{pmatrix} 5 \\ 6 \\ 0 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \\ 4 \end{pmatrix}$, for $\lambda \in \mathbb{R}$.

C. * The nullity of A is 1.

D. The nullity of A is 2.

E. The nullity of A is 3.

F. * The general solution to this system can be written as $x = \begin{pmatrix} 5 \\ 6 \\ 0 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \\ 4 \end{pmatrix}$, for

$\lambda \in \mathbb{R}$.

8. **(6 points)** Let A be an $n \times n$ matrix. Which of the following statements are equivalent to “The determinant of A is 0”?

A. The system of linear equations $Ax = 0$ has the unique solution $x = 0$.

B. The rows of A are linearly independent.

C. * A has at least one eigenvalue zero.

D. The columns of A are linearly independent.

E. * The system of linear equations $Ax = 0$ has infinitely many solutions.

F. * The rank of A is less than n .

9. (4 points) Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix}.$$

A. $A^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 1 & 0 \\ 1 & 4 & 1 \end{pmatrix}.$

B. The matrix A is not invertible.

C. $A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix}.$

D. * $A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{pmatrix}.$

E. $A^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 2 & 0 \\ -1 & -2 & 1 \end{pmatrix}.$

F. $A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ -1 & -2 & 1 \end{pmatrix}.$

10. (4 points) Compute the eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

A. There is only one eigenvalue $\lambda = 1$.

B. The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 1$.

C. * The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$.

D. The eigenvalues are $\lambda_1 = -i$ and $\lambda_2 = i$.

E. The eigenvalues are $\lambda_1 = 2 - \sqrt{2}$ and $\lambda_2 = 2 + \sqrt{2}$.

F. There is only one eigenvalue $\lambda = 2$.

11. **(6 points)** Let A be an $n \times n$ matrix. Which of the following statements are true?

- A. Any $n \times n$ matrix A is diagonalizable.
- B. * If all eigenvalues of A are distinct, then A is diagonalizable.
- C. * The trace of the matrix A is given by the sum of all eigenvalues, including their multiplicities.
- D. If A is invertible, then all eigenvalues λ have the property that $|\lambda| \leq 1$.
- E. * If A is a normal matrix, then A is diagonalizable.
- F. The trace of the matrix A is given by the product of all eigenvalues, including their multiplicities.

12. **(6 points)** Consider the matrix

$$A = \begin{pmatrix} 4 & i & i \\ -i & 4 & 5 \\ -i & 5 & 2 \end{pmatrix}.$$

Which of the following is true?

- A. A is skew-Hermitian.
- B. * A is normal.
- C. A is unitary.
- D. A is real symmetric.
- E. * A is Hermitian.
- F. A is orthogonal.

13. **(6 points)** Which of the following statements are equivalent to “ U is a unitary $n \times n$ matrix”?

- A. * All rows of U are orthonormal.
- B. * $|Ux| = |x|$ for all vectors $x \in \mathbb{C}^n$.
- C. $U^* = U$.
- D. U is normal and has eigenvalues ± 1 .
- E. * All eigenvalues of U have absolute value 1 and U is normal.
- F. U can be diagonalized.

14. (4 points) Compute the LU decomposition of the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 6 & 6 \end{pmatrix}$$

such that all diagonal entries of L are one. What are the diagonal entries of U ?

- A. * U has diagonal entries 2, 3.
- B. U has diagonal entries $-1, -2$.
- C. U has diagonal entries 1, 2.
- D. U has diagonal entries 1, 3.
- E. U has diagonal entries $-2, 3$.
- F. U has diagonal entries $-1, 1$.

15. (4 points) Consider the matrix

$$A = \begin{pmatrix} 1 & 8 \\ -1 & 2 \end{pmatrix}.$$

Which of the following is a valid QR -decomposition?

- A. $Q = \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 3i & -i \\ 0 & 4 \end{pmatrix}.$
- B. None of the options are valid QR decompositions.
- C. $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad R = \sqrt{2} \begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}.$
- D. $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad R = \sqrt{2} \begin{pmatrix} 2 & 3 \\ 0 & -3 \end{pmatrix}.$
- E. $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 8 \\ -1 & 2 \end{pmatrix}.$
- F. * $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad R = \sqrt{2} \begin{pmatrix} 1 & 3 \\ 0 & 5 \end{pmatrix}.$

16. **(6 points)** Consider the matrix

$$A = \begin{pmatrix} 2 & 2 & 2 & 2 \\ \frac{17}{10} & \frac{1}{10} & -\frac{17}{10} & -\frac{1}{10} \\ \frac{3}{5} & \frac{9}{5} & -\frac{3}{5} & -\frac{9}{5} \end{pmatrix}.$$

A has a singular value decomposition $A = U\Sigma V^*$ with

$$U = \frac{1}{5} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad V^* = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Which of the following statements are true?

- A. The singular values are 5, 4, 3.
- B. * $\text{rank}(A) = 2$.
- C. $\text{rank}(A) = 3$.
- D. * U is unitary.
- E. $\text{rank}(A) = 1$.
- F. * The singular values are 4, 3, and 0.

17. (18 points)

Use Gaussian elimination to find the general solution to the system of linear equations

$$\begin{aligned}2x_1 + 3x_3 + 10x_4 &= 3, \\x_1 + x_2 + x_3 + 6x_4 &= 1, \\x_3 + 2x_4 &= -1, \\-x_1 + 3x_3 + 4x_4 &= -6.\end{aligned}$$

(Here, you need to write down all steps of your solution in order to receive full points.)

Solution. We use the augmented matrix notation and bring the system into reduced row-echelon form. We find

$$\left(\begin{array}{cccc|c} 2 & 0 & 3 & 10 & 3 \\ 1 & 1 & 1 & 6 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ -1 & 0 & 3 & 4 & -6 \end{array} \right)$$

Take $R_1 - 2R_2 \rightarrow R_2$ and $R_1 + 2R_4 \rightarrow R_4$:

$$\left(\begin{array}{cccc|c} 2 & 0 & 3 & 10 & 3 \\ 0 & -2 & 1 & -2 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 9 & 18 & -9 \end{array} \right)$$

Take $-9R_3 + R_4 \rightarrow R_4$ and $R_2 - R_3 \rightarrow R_2$ and $R_1 - 3R_3 \rightarrow R_1$:

$$\left(\begin{array}{cccc|c} 2 & 0 & 0 & 4 & 6 \\ 0 & -2 & 0 & -4 & 2 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

Take $R_1/2 \rightarrow R_1$ and $R_2/(-2) \rightarrow R_2$:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

The general solution to this system can be written as

$$x = \begin{pmatrix} 3 \\ -1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 2 \\ -1 \end{pmatrix},$$

for $\lambda \in \mathbb{R}$.

18. (18 points)

Consider the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}.$$

Diagonalize the matrix A , and additionally compute all singular values of A . (Here, you need to write down all steps of your solution in order to receive full points.)

Solution. We first compute the eigenvalues. We find

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} 2 - \lambda & 3 \\ 0 & -1 - \lambda \end{pmatrix} = (2 - \lambda)(-1 - \lambda).$$

Hence, the eigenvalues are $\lambda_- = -1$ and $\lambda_+ = 2$. The corresponding eigenvectors are the following. For $\lambda_- = -1$, we need to solve

$$\begin{pmatrix} 3 & 3 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix}.$$

For example, let us choose $v_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. For $\lambda_+ = 2$, we need to solve

$$\begin{pmatrix} 0 & 3 & | & 0 \\ 0 & -3 & | & 0 \end{pmatrix}.$$

For example, let us choose $v_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The diagonalizing matrix V has the eigenvectors as columns, i.e.,

$$V = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Its inverse is

$$V^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Hence, a diagonalization is

$$A = V D V^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

To compute the singular values, we compute

$$A^* A = \begin{pmatrix} 4 & 6 \\ 6 & 10 \end{pmatrix}.$$

Next, we compute the eigenvalues of $A^* A$. We find

$$0 = \det(A^* A - \lambda I) = \det \begin{pmatrix} 4 - \lambda & 6 \\ 6 & 10 - \lambda \end{pmatrix} = (4 - \lambda)(10 - \lambda) - 36 = \lambda^2 - 14\lambda + 4.$$

Hence, the eigenvalues are $\lambda_{\pm} = 7 \pm \sqrt{45}$. The singular values $\sigma_{\pm}$ are the square roots of these eigenvalues, hence they are $\sigma_{\pm} = \sqrt{7 \pm \sqrt{45}}$.

