Week 4: Systems of Linear Equations, Gaussian Elimination

1. MULTI Single

   Solve the following system of linear equations:

$$\begin{aligned} x_1 + 3\,x_2 - 5\,x_3 &= 4 \\ x_1 + 4\,x_2 - 8\,x_3 &= 7 \\ -3\,x_1 - 7\,x_2 + 9\,x_3 &= -6 \end{aligned}$$

   (a)
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 \\ 6 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

   (b)
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 \\ 6 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$$

   (c)
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$$

   (d)
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}$$

2. MULTI Single

   Find $\alpha \in \mathbb{R}$ such that following system of linear equations has infinitely many solutions:

$$\begin{aligned} 3x_1 + (6+\alpha)x_2 &= 11 \\ x_1 + 2x_2 &= 3 \end{aligned}$$

   (a) $\alpha = 1$
   (b) There exists no such α.
   (c) $\alpha = 2$
   (d) $\alpha = 0$

3. MULTI Single

   Which of the following is true for homogeneous systems of linear equations?

   (a) If $\vec{a}$ is a solution, $\exists k \in \mathbb{R}$ such that $k\vec{a}$ is not a solution
   (b) If $\vec{a}$ and $\vec{b}$ are both solutions, then $\vec{a} + \vec{b}$ is also a solution
   (c) The system might not have a solution
   (d) We can always find a solution $\vec{a}$ such that all its components a_i are positive

4. MULTI Single

Let $\vec{a}$ and $\vec{b}$ be both solutions to a system of linear equations $A\vec{x} = \vec{v}$. When is $\vec{a}+\vec{b}$ also a solution?

(a) Always
(b) Never
(c) When $\vec{v} = 0$
(d) When $\vec{v} \neq 0$

5. MULTI Single

Suppose the homogeneous system of linear equations $Av = 0$ has the **unique** solution $v = 0$. Let $b \neq 0$. Then $Ax = b$:

(a) might have infinitely many solutions.
(b) has a unique solution.
(c) might not have a solution.
(d) might have exactly two solutions.

6. MULTI Multiple

Consider some 2×5 matrix A, and some vector $b \in \mathbb{R}^2$. Then the system of linear equations $Ax = b$ might have

(a) no solution.
(b) exactly two solutions.
(c) infinitely many solutions.
(d) exactly one solution.

7. MULTI Single

Consider the standard basis in $\mathbb{R}^3$: $\{e_x, e_y, e_z\}$.
Which of the following matrices represents a counterclockwise rotation with angle φ around the z-axis?

(a) $\mathcal{R} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 \end{bmatrix}$
(b) $\mathcal{R} = \begin{bmatrix} 1 & 0 & -\sin\varphi \\ \cos\varphi & 1 & \cos\varphi \\ \sin\varphi & 0 & 1 \end{bmatrix}$
(c) $\mathcal{R} = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix}$
(d) $\mathcal{R} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. MULTI Single

Consider the vector space $P_2(\mathbb{R}) = \{p(x) \mid p(x) \text{ is a quadratic polynomial}\}$.
Is the derivative operator $\mathcal{D} : p(x) \mapsto p'(x) \equiv \dfrac{\mathrm{d}}{\mathrm{d}x} p(x)$ a linear operator? If it is, how

is it represented in the standard basis $\mathfrak{B} = \{1, x, x^2\}$?

Hint: You can express a polynomial $ax^2 + bx + c$ *as* $\begin{bmatrix} c \\ b \\ a \end{bmatrix}$

(a) $\mathcal{D}$ is a linear operator with $[\mathcal{D}]_{\mathfrak{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\mathcal{D}$ is a linear operator with $[\mathcal{D}]_{\mathfrak{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(c) $\mathcal{D}$ is a linear operator with $[\mathcal{D}]_{\mathfrak{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\mathcal{D}$ is not a linear operator

9. MULTI Single

Which of the following is equivalent to $(A \cdot B \cdot C)^T$

(a) $C^T \cdot A^T \cdot B^T$
(b) $A^T \cdot B^T \cdot C^T$
(c) $C^T \cdot B^T \cdot A^T$
(d) $B^T \cdot C^T \cdot A^T$

10. MULTI Single

Let A be a (3×4) matrix, and B be a matrix such that $A^T \cdot B$ and $B \cdot A^T$ are both defined. What are the dimensions of B

(a) (3×4)
(b) (4×3)
(c) (4×4)
(d) (3×3)

Total of marks: 10