Week 8: Eigenvalues and eigenspaces

1. MULTI Single

   A 5×5 matrix has eigenvalues $\lambda_1, \ldots, \lambda_5$. If $\lambda_1 = 2+i$, $\lambda_2 = \frac{i}{\sqrt{2}}$, $\lambda_3 = 2$ then what are the values of λ_4 and λ_5?

   (a) $\lambda_4 = \lambda_5 = 0$
   (b) $\lambda_4 = 2+i, \lambda_5 = \frac{i}{\sqrt{2}}$
   (c) The information given is insufficient to determine λ_4 and λ_5
   (d) $\lambda_4 = 2-i, \lambda_5 = -\frac{i}{\sqrt{2}}$

2. MULTI Single

   Let A be an $n \times n$ matrix that has eigenvalues $1, 3, 5, \ldots, 2n-1$. Compute $\mathrm{tr}(A)$ and $\det(A)$. (Recall that $n! := 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$.)

   (a) $\mathrm{tr}(A) = n^2, \det(A) = \frac{(2n)!}{2(n!)}$
   (b) $\mathrm{tr}(A) = n^2, \det(A) = \frac{(2n)!}{2^n(n!)}$
   (c) $\mathrm{tr}(A) = n, \det(A) = \frac{(2n)!}{n!}$
   (d) $\mathrm{tr}(A) = (n+1)^2, \det(A) = \frac{(2n+2)!}{2^n((n+1)!)}$

3. MULTI Single

   Let A be a matrix such that

   $$\det(A - \lambda I) = -\lambda^3(\lambda - 1)(2\lambda + 1)^2.$$

   Compute $\mathrm{tr}(A)$ and $\det(A)$.

   (a) $\mathrm{tr}(A) = 0, \det(A) = -1$
   (b) $\mathrm{tr}(A) = 0, \det(A) = 0$
   (c) $\mathrm{tr}(A) = \frac{1}{2}, \det(A) = 0$
   (d) $\mathrm{tr}(A) = \frac{1}{2}, \det(A) = -\frac{1}{2}$

4. MULTI Single

   Let A be a 7×7 matrix such that

   $$\det(A - \lambda I) = (\lambda - 2 + i)(\lambda - i)(\lambda - \sqrt{2})^2(\lambda - 1)q(\lambda)$$

   where $q(\lambda)$ is some polynomial. Compute $\mathrm{tr}(A)$ $\det(A)$.

   (a) $\mathrm{tr}(A) = 2 + 2\sqrt{2}, \det(A) = 2 + 4i.$
   (b) Information given in insufficient to determine $\mathrm{tr}(A)$ and $\det(A)$.
   (c) $\mathrm{tr}(A) = 5 + 2\sqrt{2}, \det(A) = 10.$
   (d) $\mathrm{tr}(A) = 2 + \sqrt{2}, \det(A) = \sqrt{2} + 2\sqrt{2}i.$

5. MULTI Single

Consider the eigenvalues of the matrix

$$S = \begin{bmatrix} 1 & 2 & 2 & 3 & 4 & 1 \\ 2 & 4 & 3 & 5 & 5 & 7 \\ 2 & 3 & 9 & 1 & 1 & 0 \\ 3 & 5 & 1 & 0 & 1 & 2 \\ 4 & 5 & 1 & 1 & 3 & 1 \\ 1 & 7 & 0 & 2 & 1 & 8 \end{bmatrix}.$$

How many eigenvalues of the matrix are **not** real?

(a) 7
(b) 0
(c) 1
(d) 3

6. MULTI Single

Let A be a matrix with characteristic polynomial

$$p(\lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4)(\lambda - 5).$$

What is the dimension of $\ker(A)$?

(a) 3
(b) 2
(c) 0
(d) 1

7. MULTI Single

Consider the matrix

$$A = \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}.$$

Compute the set of tuples $(\lambda, a_\lambda, g_\lambda)$ where λ is an eigenvalue, a_λ its algebraic multiplicity and g_λ is its geometric multiplicity.

(a) $(\lambda_1 = 0, a_1 = 2, g_1 = 2)$
(b) $(\lambda_1 = 1, a_1 = 1, g_1 = 2)$
(c) $(\lambda_1 = 1, a_1 = 2, g_1 = 2)$
(d) $(\lambda_1 = 1, a_1 = 2, g_1 = 1)$

8. MULTI Single

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 3 & 0 \\ 3 & 2 & 2 \end{bmatrix}.$$

Compute the set of tuples $(\lambda, a_\lambda, g_\lambda)$ where λ is an eigenvalue, a_λ its algebraic multiplicity and g_λ is its geometric multiplicity.

(a) $(\lambda_1 = 1, a_1 = 1, g_1 = 0), (\lambda_2 = 2, a_2 = 1, g_2 = 0), (\lambda_3 = 3, a_3 = 1, g_3 = 0)$

(b) $(\lambda_1 = 1, a_1 = 0, g_1 = 1), (\lambda_2 = 2, a_2 = 0, g_2 = 1), (\lambda_3 = 3, a_3 = 0, g_3 = 1)$
(c) $(\lambda_1 = 1, a_1 = 1, g_1 = 1), (\lambda_2 = 2, a_2 = 1, g_2 = 1), (\lambda_3 = 3, a_3 = 1, g_3 = 1)$
(d) $(\lambda_1 = 1, a_1 = 3, g_1 = 2)$

9. MULTI Single

A matrix A has characteristic polynomial

$$p(\lambda) = \lambda(\lambda + 2)^2 + 3\lambda - 1.$$

Use the Cayley-Hamilton theorem to deduce a formula for A^{-1} in terms of A.

(a) A is not invertible
(b) $A^{-1} = A$
(c) $A^{-1} = (A + 2)^2 + 3I$
(d) $A^{-1} = A(A + 2)^2 + 3A - I$

10. MULTI Single

Consider the matrix

$$A = \begin{bmatrix} 1/2 & 10 & 0 \\ 0 & 1/2 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix}$$

Which of the following statements is false?

(a) A^4 has an eigenvalue $1/4$.
(b) A^{-1} has an eigenvalue 2.
(c) A is invertible.
(d) A^4 has an eigenvalue 1.

Total of marks: 10