Week 10: Hermitian/real symmetric and unitary/orthogonal matrices

(1) MULTIPLE CHOICE One answer only

A matrix H is called *Hermitian* if $H = H^\dagger$. A matrix A is called *anti-Hermitian* if $A = -A^\dagger$. It is possible to write any matrix as a sum of a Hermitian and an anti-Hermitian matrix. Let

$$C = \begin{bmatrix} 1 & -i \\ 2i & 3 \end{bmatrix}$$

and find a Hermitian matrix H and an anti-Hermitian matrix A such that $C = H + A$.

a. $H = \begin{bmatrix} 1 & \frac{3}{2}i \\ -\frac{3}{2}i & 3 \end{bmatrix}$ $A = \begin{bmatrix} 0 & -\frac{i}{2} \\ -\frac{i}{2} & 0 \end{bmatrix}$

b. $H = \begin{bmatrix} 0 & \frac{i}{2} \\ \frac{i}{2} & 0 \end{bmatrix}$ $A = \begin{bmatrix} 1 & -\frac{3}{2}i \\ \frac{3}{2}i & 3 \end{bmatrix}$

c. $H = \begin{bmatrix} 0 & -\frac{i}{2} \\ -\frac{i}{2} & 0 \end{bmatrix}$ $A = \begin{bmatrix} 1 & \frac{3}{2}i \\ -\frac{3}{2}i & 3 \end{bmatrix}$

d. $H = \begin{bmatrix} 1 & -\frac{3}{2}i \\ \frac{3}{2}i & 3 \end{bmatrix}$ $A = \begin{bmatrix} 0 & \frac{i}{2} \\ \frac{i}{2} & 0 \end{bmatrix}$

(2) MULTIPLE CHOICE One answer only

Find the characteristic polynomial of the matrix

$$H = \begin{bmatrix} -1 & 1-2i & 0 \\ 1+2i & 0 & -i \\ 0 & i & 1 \end{bmatrix}.$$

Find this polynomial explicitly and determine the number of real roots.

a. The characteristic polynomial $-\lambda^3 + 7\lambda - 4$ has two real roots
b. The characteristic polynomial $-\lambda^3 + 7\lambda - 4$ has three real roots
c. The characteristic polynomial $-\lambda^3 + 7\lambda - 4$ has only one real root
d. The characteristic polynomial $-\lambda^3 - 4$ has three real roots

(3) MULTIPLE CHOICE One answer only

The matrix

$$A = \begin{bmatrix} i & 2+i & 3 \\ -2+i & 2i & -1 \\ -3 & 1 & 3i \end{bmatrix}$$

is

a. Unitary

b. Skew-Hermitian
c. Hermitian
d. Orthogonal

(4) Multiple choice | One answer only

Let U be unitary, and define the matrix

$$A = U^* \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} U.$$

Is A positive definite?

a. For some U it is, for others not.
b. No.
c. Yes.

(5) Multiple choice | One answer only

A normal matrix U is called *unitary* if $UU^\dagger = I$. Is the matrix

$$U = \frac{1}{2} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

normal and is it unitary?

a. U is not normal and U is not unitary
b. U is normal and U is unitary
c. U is normal but U is not unitary
d. U is not normal but U is unitary

(6) Multiple choice | One answer only

A normal matrix U is called *unitary* if $UU^\dagger = I$. Is the matrix

$$U = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

normal and is it unitary?

a. U is not normal but U is unitary
b. U is normal but U is not unitary
c. U is not normal and U is not unitary
d. U is normal and U is unitary

(7) Multiple choice | One answer only

A normal matrix U is called *unitary* if $UU^\dagger = I$. Is the matrix

$$U = \frac{1}{2} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

normal and is it unitary?

a. U is normal but U is not unitary
b. U is not normal but U is unitary
c. U is not normal and U is not unitary
d. U is normal and U is unitary

(8) Multiple choice One answer only

Consider the matrix

$$U = \begin{bmatrix} i & 0 & 0 \\ 0 & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$$

and the vector

$$x = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}.$$

Compute the length of Ux, i.e., compute $|Ux|$.

a. $|Ux| = \sqrt{39}$.
b. $|Ux| = 1$.
c. $|Ux| = \sqrt{28}$.
d. $|Ux| = \sqrt{17}$.

(9) Multiple choice One answer only

Let U_1 and U_2 both be unitary $n \times n$ matrices. Then the product U_1U_2 is

a. unitary
b. orthogonal
c. real symmetric
d. Hermitian

(10) Multiple choice One answer only

Let Q be an orthogonal 5×5 matrix. Then

a. At least one eigenvalue must have non-zero imaginary part.
b. All eigenvalues of Q are either $+1$ or -1.
c. All eigenvalues of Q are real.
d. Q must have an eigenvalue $+1$ or -1.

Total of marks: 10