Constructor University
Fall 2025

December 5, 2025

Elements of Linear Algebra
Practice Final Exam

Instructions:

- The exam has 16 multiple choice questions (several answers can be correct!) and 2 longer questions. The total number of points is 118.
- For the multiple choice questions, it is sufficient to mark the final answer(s) only. (No solution steps necessary.) There are no negative points, but of course there are fewer points if wrong answers are selected, or if right answers are not selected.
- For the longer exercises 17 and 18, you need to show your work, i.e., carefully write down the steps of your solution. You will receive points not just based on your final answer, but on the correct steps in your solution.
- No tools or other resources are allowed for this exam. In particular, no notes and no calculators.
- You are free to refer to any results proven in class or the homework sheets unless stated otherwise (and unless the problem is to reproduce a result from class or the homework sheets).

Code of Academic Integrity

I pledge that the answers of this exam are my own work without the assistance of others or the usage of unauthorized material or information.

Sign to confirm that you adhere to the Academic Integrity Code:

Name: ____

Signature: ____________________________________

Matric./Student No.: __________________________

1. **(6 points)** Consider the function

$$f_\lambda(x) = x^2 + 4\lambda x - 4\lambda,$$

with parameter $\lambda \in \mathbb{R}$. Which of the following is true?

A. For $\lambda = 0$, the equation $f_\lambda(x) = 0$ has one real solution.

B. The domain of $f_\lambda(x)$ is $\mathbb{R}$.

C. For $\lambda < -1$, the equation $f_\lambda(x) = 0$ has two complex solutions with non-vanishing imaginary part.

D. For $\lambda = -1$, the equation $f_\lambda(x) = 0$ has two complex solutions with non-vanishing imaginary part.

E. For $-1 < \lambda < 0$, the equation $f_\lambda(x) = 0$ has two complex solutions with non-vanishing imaginary part.

F. The domain of $f_\lambda(x)$ is the interval $[\lambda^2 + \lambda, \infty)$.

2. **(6 points)** Consider the vectors

$$a = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}.$$

Compute their scalar (dot) product, their cross product, and the lengths of the vectors.

A. The cross product is $a \times b = \begin{pmatrix} 0 \\ -7 \\ -7 \end{pmatrix}$.

B. The length of a is $|a| = \sqrt{11}$ and the length of b is $|b| = 3$.

C. The scalar (dot) product is $a \cdot b = 1$.

D. The cross product is $a \times b = \begin{pmatrix} 4 \\ 6 \\ -4 \end{pmatrix}$.

E. The scalar (dot) product is $a \cdot b = -1$.

F. The length of a is $|a| = 3$ and the length of b is $|b| = 1$.

3. **(4 points)** Consider the set

$$P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + 3y - 4z = 0 \right\} \subset \mathbb{R}^3.$$

Note that P describes a plane through the origin, and is thus a vector space itself. Which of the following is a basis for P? Note that only one answer is correct here.

A. $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$.

B. $\begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 4 \end{pmatrix}$.

C. $\begin{pmatrix} 1 \\ 2 \\ -7 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$.

D. $\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$.

E. $\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 6 \\ 14 \end{pmatrix}$.

F. $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$.

4. **(4 points)** Recall that A^T denotes the transpose of the matrix A. Now let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \qquad \text{and} \quad v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Compute $A^T v$.

A. $A^T v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

B. $A^T v = \begin{pmatrix} 4 \\ 5 \\ 8 \end{pmatrix}$.

C. $A^T v = 17$.

D. $A^T v = \begin{pmatrix} 4 \\ 2 \\ 8 \end{pmatrix}$.

E. $A^T v = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$.

F. $A^T v = \begin{pmatrix} 6 \\ 5 \\ 6 \end{pmatrix}$.

5. **(4 points)** Let
$$A = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix},\ B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},\ C = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}.$$
Calculate the matrix product ABC.

A. $\begin{pmatrix} 3 & 6 \\ 12 & 18 \end{pmatrix}$.

B. $\begin{pmatrix} 12 & 12 \\ 10 & 10 \end{pmatrix}$.

C. $\begin{pmatrix} 6 & 10 \\ 10 & 6 \end{pmatrix}$.

D. $\begin{pmatrix} 12 & 10 \\ 12 & 10 \end{pmatrix}$.

E. $\begin{pmatrix} 12 & 6 \\ 6 & 12 \end{pmatrix}$.

F. $\begin{pmatrix} 6 & 12 \\ 6 & 18 \end{pmatrix}$.

6. **(4 points)** Which of the following describes the solution(s) to the system of linear equations
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4, \\ x_2 + 2x_3 &= 3, \\ -x_2 + 2x_3 &= 1. \end{aligned}$$

A. The system of equations has no solutions.

B. The unique solution is $x = (1, 1, 1)$.

C. The unique solution is $x = (0, 1, 1)$.

D. The system of equations has infinitely many solutions $x = (1, 0, 0) + \lambda(1, 1, 1)$.

E. The system of equations has infinitely many solutions $x = (2, 1, 0) + \lambda(1, 2, 1)$.

F. The unique solution is $x = (2, 1, 1)$.

7. **(6 points)** A system of linear equations $Ax = b$ has been brought, through Gaussian elimination, into the reduced row-echelon form (in augmented matrix notation)

$$\left(\begin{array}{cccc|c} 1 & 3 & 3 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 4 & 1 & 9 \end{array}\right).$$

Which of the following statements are true?

A. The nullity of A is 2.

B. The general solution is $x = \begin{pmatrix} 7 \\ 0 \\ 0 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 0 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ -1 \\ 4 \end{pmatrix}$, for $\lambda, \mu \in \mathbb{R}$.

C. The general solution is $x = \begin{pmatrix} 7 \\ 0 \\ 0 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 0 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ 0 \\ 4 \end{pmatrix}$, for $\lambda, \mu \in \mathbb{R}$.

D. The nullity of A is 1.

E. The general solution is $x = \begin{pmatrix} 7 \\ 0 \\ 0 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 0 \\ 6 \end{pmatrix}$, for $\lambda \in \mathbb{R}$.

F. The nullity of A is 3.

8. **(6 points)** Let A be an $n \times n$ matrix. Which of the following statements are equivalent to "The rank of A is n"?

A. A has at least one eigenvalue zero.

B. The system of linear equations $Ax = 0$ has infinitely many solutions.

C. The system of linear equations $Ax = 0$ has the unique solution $x = 0$.

D. The columns of A are linearly independent.

E. A is invertible.

F. The determinant of A is zero.

9. **(4 points)** Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ 2 & 4 & 0 \end{pmatrix}.$$

A. $A^{-1} = \begin{pmatrix} -1 & -2 & 4 \\ 1 & 1 & -2 \\ 1 & 3 & 4 \end{pmatrix}.$

B. The matrix A is not invertible.

C. $A^{-1} = \begin{pmatrix} -2 & -2 & -\frac{3}{2} \\ 1 & 1 & 1 \\ 1 & 5 & -8 \end{pmatrix}.$

D. $A^{-1} = \begin{pmatrix} -1 & -2 & 4 \\ 1 & 1 & -2 \\ 1 & 5 & -8 \end{pmatrix}.$

E. $A^{-1} = \begin{pmatrix} -1 & -2 & 4 \\ -1 & -1 & 2 \\ -1 & 3 & 4 \end{pmatrix}.$

F. $A^{-1} = \begin{pmatrix} -2 & -2 & -\frac{3}{2} \\ 1 & 1 & 1 \\ -1 & 2 & 4 \end{pmatrix}.$

10. **(4 points)** Compute the eigenvalues of the matrix

$$A = \begin{pmatrix} 5 & 1 \\ -3 & 1 \end{pmatrix}.$$

A. The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 4$.

B. The eigenvalues are $\lambda_1 = -i$ and $\lambda_2 = i$.

C. There is only one eigenvalue $\lambda = 3$.

D. The eigenvalues are $\lambda_1 = 3 - \sqrt{2}$ and $\lambda_2 = 3 + \sqrt{2}$.

E. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$.

F. There is only one eigenvalue $\lambda = 1$.

11. **(6 points)** Let A be an $n \times n$ matrix. Which of the following statements are true?

    A. If A has only real entries, then all eigenvalues are real.

    B. A is diagonalizable if and only if there exists a Hermitian matrix H and a diagonal matrix D such that $A = DHD^*$.

    C. The trace of the matrix A is given by the product of all eigenvalues, including their multiplicities.

    D. The trace of the matrix A is given by the sum of all eigenvalues, including their multiplicities.

    E. If λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k for any $k \in \mathbb{N}$.

    F. A is diagonalizable if and only if the algebraic multiplicity equals the geometric multiplicity for every eigenvalue.

12. **(6 points)** Consider the matrix

$$A = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

    Which of the following is true?

    A. A is normal.

    B. A is orthogonal.

    C. A is Hermitian.

    D. A is real symmetric.

    E. A is unitary.

    F. A is skew-Hermitian.

13. **(6 points)** Which of the following statements are equivalent to "U is a unitary $n \times n$ matrix"?

    A. $U^* = U$.

    B. All eigenvalues of U have absolute value 1 and U is normal.

    C. $|Ux| = |x|$ for all vectors $x \in \mathbb{C}^n$.

    D. All eigenvalues of U are real.

    E. $U^* = U^{-1}$.

    F. $U^*U = UU^*$.

14. **(4 points)** Compute the LU decomposition of the matrix

$$A = \begin{pmatrix} -2 & 1 \\ -8 & -1 \end{pmatrix}$$

such that all diagonal entries of L are one. What are the diagonal entries of U?

A. U has diagonal entries $-2, 3$.

B. U has diagonal entries $1, 2$.

C. U has diagonal entries $-1, 1$.

D. U has diagonal entries $-2, -5$.

E. U has diagonal entries $-1, -2$.

F. U has diagonal entries $2, 5$.

15. **(6 points)** Which of the following statements is true?

A. Only real square matrices have QR decompositions.

B. Only invertible real square matrices have QR decompositions.

C. Every invertible real square matrix has a QR decomposition.

D. Every real square matrix has a QR decomposition.

E. Every real $m \times n$ matrix with $m > n$ has a QR decomposition.

F. If a matrix has a QR decomposition, then its determinant must be ± 1.

16. **(6 points)** Consider the matrix

$$A = \begin{pmatrix} 2 & 2 & 2 & 2 \\ \frac{3}{5} & \frac{3}{5} & -\frac{3}{5} & -\frac{3}{5} \\ \frac{4}{5} & \frac{4}{5} & -\frac{4}{5} & -\frac{4}{5} \end{pmatrix}.$$

A has a singular value decomposition $A = U\Sigma V^*$ with

$$U = \frac{1}{5}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad V^* = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Which of the following statements are true?

A. U is unitary.

B. $\text{rank}(A) = 3$.

C. The singular values are 5, 4, 3.

D. $\text{rank}(A) = 2$.

E. The singular values are 4, 2, and 0.

F. $\text{rank}(A) = 1$.

17. **(18 points)**

Use Gaussian elimination to find the general solution to the system of linear equations

$$\begin{aligned} x_1 - x_2 - x_3 &= -1, \\ -2x_2 - 4x_4 &= -6, \\ x_1 + x_2 + x_3 + 6x_4 &= 9, \\ x_1 + x_3 + 4x_4 &= 6. \end{aligned}$$

(Here, you need to write down all steps of your solution in order to receive full points.)

18. **(18 points)**

Consider the matrix
$$A = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}.$$
Diagonalize the matrix A, and additionally compute all singular values of A. (Here, you need to write down all steps of your solution in order to receive full points.)