

Stochastic Modeling and Financial Mathematics

Quiz 6

Name: _____

1. **(8 points)** Recall that an Itô process $X(t)$ is the solution to the stochastic differential equation (SDE)

$$dX(t) = f(X(t), t) dt + g(X(t), t) dW(t).$$

Recall that for functions $F(X(t), t)$, Itô's lemma says that

$$dF = \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} f + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} g^2 \right) dt + g \frac{\partial F}{\partial x} dW.$$

- (a) How does Itô's lemma simplify if $X(t) = W(t)$, i.e., $X(t)$ is just Brownian motion?
(b) Now consider $X(t) = W(t)$ and

$$S(W(t), t) = e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}.$$

What SDE does this S satisfy?

- (c) Now consider $X(t) = S(t)$ and the function $F(S(t), t) = S(t)^2$. What SDE does this F satisfy? What is the expectation value of $F(t)$?

2. (6 points) We consider the following python program.

```
from pylab import *

T = 1.0
N = 500
dt = T/N
M = 4000
dW = normal(0, 1, (M,N))*sqrt(dt)
W = c_[zeros(M), cumsum(dW, axis=1)]

mu = 0.7
sigma = 0.4
Y = c_[ones(M), cumprod(exp(dt*(mu-0.5*sigma**2) + sigma*dW), axis=1)]

Ymean = mean(Y, axis=0)
Ystd = std(Y, axis=0)

Yplot = Y[:6,:].T
t = linspace(0,T,N+1)
figure()
plot (t,Ymean, 'b',
      t,Ymean+Ystd, 'r',
      t,Yplot, 'k:',
      t,Ymean-Ystd, 'r')
show()
```

Draw the output of this program.

3. **(6 points)** Note that the function $F(S(t), t) = (1 + t)S(t)$ satisfies the SDE

$$dF = \left(\mu + \frac{1}{1+t} \right) F dt + \sigma F dW(t),$$

with $S(t)$ geometric Brownian motion (with parameters μ, σ). Now consider the following python program:

```
from pylab import *

T = 1.0
N = 20000
tt, dt = linspace(0,T,N+1,retstep=True)

mu = 0.5
sigma = 3.0

dW = normal(0, sqrt(dt), N)
W = r_[0,cumsum(dW)]
S = exp((mu-sigma**2/2)*tt + sigma*W)

F = zeros(N+1)
F[0] = S[0]

#Euler Maruyama
for i in range(N):
    ???

Fexact = (1+tt)*S

figure()
plot(tt, F, 'bo',
      tt, Fexact, 'g-')
show()
```

- Briefly (one or two sentences) describe what this python program is doing.
- Add the right code in the “for” loop (???) to make the Euler-Mayurama method work.