Prof. Sören Petrat, Constructor University
Lecture notes from Fall 2025

Internal Rate of Return (IRR) or yield:

Given a cash flow C_i and n (number of years), the present value is

$$PV(r) = \sum_{i=1}^{n} \frac{C_i}{(1+r)^i} .$$

But often such financial instruments have a certain price P.

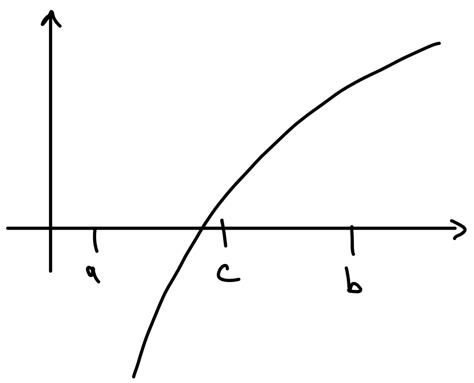
We want to know which r solves $PV(r) = P$. This r we call IRR.

Sometimes one defines the net-present-value $NPV(r) = PV(r) - P$.

$\Rightarrow$ Here the zeroes of $NPV(r)$ give us the IRR.

Root Finding Algorithms:

- **Bisection:**



- choose $a < b$ such that $f(a)f(b) < 0$

  (if $f(a)f(b) = 0$ then we're done: a or b is the root)

- set $c = \frac{a+b}{2}$

  ↳ if $f(c) = 0 \Rightarrow$ done

  ↳ if $f(a)f(c) < 0 \Rightarrow$ root in $[a,c]$

  ↳ if $f(c)f(b) < 0 \Rightarrow$ root in $[c,b]$

- repeat with either $[a,c]$ or $[c,b]$

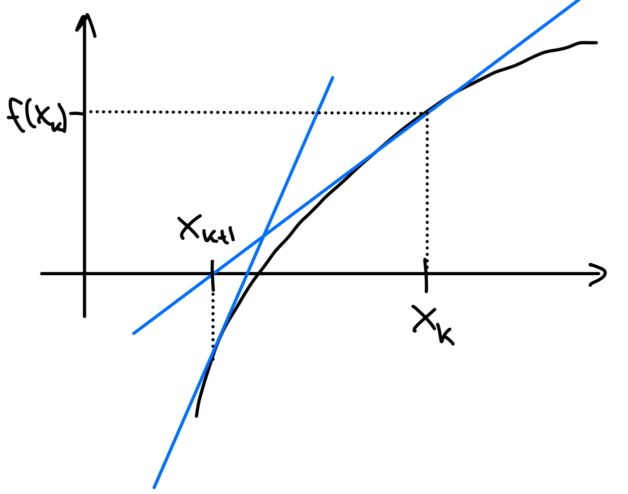
- Advantages: • robust method, only continuity of f necessary (except if $f(x) \geq 0 \ \forall x$)

- Disadvantages: • roots where $f(x) \geq 0$ (or ≤ 0) in some region around x cannot be found

  • very slow: error after n+1 steps $\varepsilon_{n+1} = \frac{1}{2}\varepsilon_n$ error after n steps

  $\Rightarrow$ on the r.h.s. we have $\varepsilon_n^{1} \Rightarrow$ linear convergence

- **Newton's Method** (Newton-Raphson method)



- we start by choosing some x_k

- then $f'(x_k) = \dfrac{f(x_k)}{x_k - x_{k+1}}$

$$\Rightarrow x_k - x_{k+1} = \frac{f(x_k)}{f'(x_k)}$$

$$\Rightarrow x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

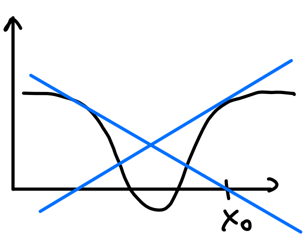
↓

Iteration

- Advantages: fast: quadratic convergence (see below)

- Disadvantages:
  - need differentiable f
  - need explicit expression for derivative f' (or need extra numerical computation)
  - it might not work!

    ↳ possible problems:
    - if f'' not continuous then convergence rate will be worse than quadratic
    - $f'(x_k) = 0$ for some k
    - x_0 too far away from the zero
    - cyclic behavior:



What is the rate of convergence?

Use Taylor expansion around x_k:

$$f(z) = f(x_k) + f'(x_k)(z-x_k) + \frac{f''(x_k)}{2}(z-x_k)^2 + \underbrace{O\left((z-x_k)^3\right)}_{R \text{ (rest)}}$$

let z be the root, i.e., $f(z)=0$

$$\Rightarrow \quad 0 = f(x_k) + f'(x_k)(z-x_k) + \frac{f''(x_k)}{2}(z-x_k)^2 + R$$

$$x_k = x_{k+1} + \frac{f(x_k)}{f'(x_k)}$$

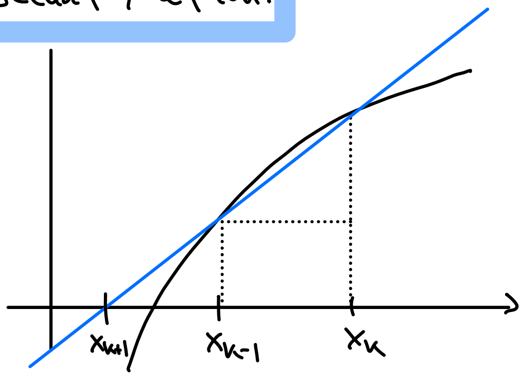
$$\Rightarrow 0 = f(x_k) + f'(x_k)\left(z - x_{k+1} - \frac{f(x_k)}{f'(x_k)}\right) + \frac{f''(x_k)}{2}(z-x_k)^2 + \underbrace{R}_{\text{neglect}}$$

$$\Rightarrow \quad z - x_{k+1} \approx -\frac{f''(x_k)}{2f'(x_k)}(z-x_k)^2$$

$$\Rightarrow \text{ error after } k+1 \text{ steps } \quad \varepsilon_{k+1} = |z - x_{k+1}| \approx \underbrace{\left|\frac{f''(x_k)}{2f'(x_k)}\right|}_{\text{if this stays indeed bounded}} \varepsilon_k^{2}$$

$\Rightarrow$ quadratic convergence!

• **Secant Method:**



– take secants instead of tangents

intercept thm. (Thales, "Strahlensatz"):

$$\frac{f(x_k)}{x_k - x_{k+1}} = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

$$\Rightarrow x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$ is the iteration

– Advantages: • still relatively fast, rate of convergence is the golden ratio ≈ 1.62
• derivative not needed

– otherwise very similar to Newton

Pythons built-in fct.: $\boxed{\text{brentq}}$

– combines advantages of several methods

– always converges for continuous functions

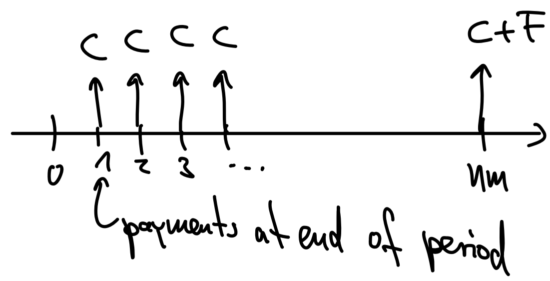
$\Rightarrow$ robust and relatively fast

1.3 Bonds

$\boxed{\text{Bond issuer (borrow)}}$ makes regular payments and a final payment to the $\boxed{\text{bond holder}}$ (buyer, lender).

↳ long-term debt, e.g., issued by governments, but also companies
↳ bond are fully repaid (= contract is finished) at $\boxed{\text{maturity date}}$

cashflow for $\boxed{\text{level-coupon bond:}}$



$$\text{present value} = \text{price of bond} = P = \left(\sum_{i=1}^{nm} \frac{C}{(1+\frac{r}{m})^i}\right) + \frac{F}{(1+\frac{r}{m})^{nm}}$$

Notation/terminology:

- C = coupon payment
- F = par value
- n = # of periods (usually years)
- m = # of payments in a period
- usually one writes $C = \frac{Fc}{m}$, with c = coupon rate

$$\Longrightarrow P = F\left(\sum_{i=1}^{nm} \frac{c/m}{(1+\frac{r}{m})^i} + \frac{1}{(1+\frac{r}{m})^{nm}}\right)$$

- P, C (or c), F, n, m determine the "bond contract".
  Given these values, one can compute the IRR r (i.e., yield (to maturity)).

Ex.: 20 year, 9% bond, $\underbrace{\text{BEY}}$, interest rate of $r = 8\%$

"bond equivalent yield" $\Rightarrow m = 2$

$$\Rightarrow \text{price } P = F\left(\sum_{i=1}^{40} \frac{0.045}{(1.04)^i} + \frac{1}{(1.04)^{40}} \right) = 1.099\, F$$

$\Rightarrow$ this bond sells at 109.9% of par.

E.g., par value $F = 1000\,\$ \Rightarrow$ price $P = 1099\,\$$

and $C = \frac{Fc}{m} = \frac{90}{2} = 45\,\$$.

Note: Often one just considers zero-coupon bonds: $c = 0$, and $P = \frac{F}{(1+r)^n}$ $(m=1)$

Note: Using geometric series $\sum_{i=0}^{N} x^i = \frac{1-x^{N+1}}{1-x}$, we find:

$$\left[(1-x)\sum_{i=0}^{N} x^i = \sum_{i=0}^{N} x^i - \sum_{i=1}^{N+1} x^i = 1 - x^{N+1} \right]$$

(For simplicity, let us take $m = 1$)

$$P = F\left(c \underbrace{\sum_{i=1}^{n} \frac{1}{(1+r)^i}} + \frac{1}{(1+r)^n} \right)$$

$$= -1 + \frac{1-(1+r)^{-n-1}}{1-(1+r)^{-1}} = \frac{\cancel{-1}+(1+r)^{-1} \cancel{+1}-(1+r)^{-n-1}}{1-(1+r)^{-1}} = \frac{1-(1+r)^{-n+1-1}}{1+r-1}$$

$$= \left(\frac{(1+r)^{-1}-(1+r)^{-n-1}}{1-(1+r)^{-1}} \right) \frac{(1+r)}{(1+r)}$$

$$= \frac{1-(1+r)^{-n}}{r}$$

$$\Rightarrow P = F\left[\frac{c}{r}\left(1-(1+r)^{-n}\right) + (1+r)^{-n}\right]$$

$$= F\left[\frac{c}{r} + \frac{1-\frac{c}{r}}{(1+r)^{n}}\right]$$

Terminology:
- If $c = r$, then $P = F$: "the bond sells at par"
- If $c > r$, then $P > F$: "the bond sells above par"
- If $c < r$, then $P < F$: "the bond sells below par"