Stochastic Modeling and Financial Mathematics — Week 5

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Lecture notes from Fall 2025

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last time: binary model

$$\begin{tikzcd}[row sep=small] & Su \\ S \arrow[ur, "p_s"] \arrow[dr, "1-p_s"'] & \\ & Sd \end{tikzcd}$$

Recall example: $K = 3000\,\$$ (call)

$$\begin{tikzcd}[row sep=small] & 4000\,\$ \quad (u=1.6) \quad \to \text{payoff: } 1000\,\$ \\ S = 2500\,\$ \arrow[ur, "p_s = \frac{1}{2}"] \arrow[dr, "1-p_s = \frac{1}{2}"'] & \\ & 2000\,\$ \quad (d=0.8) \quad \to \text{payoff: } 0\,\$ \end{tikzcd}$$

We dismissed the idea of option price = average w.r.t. stock market probabilities (from our model) because of the possibility of making risk-free profit.

New idea: Option price (value) = cost of a portfolio that leads to no risk-free profit for seller (meaning seller meets obligation exactly).
Such a portfolio is called **replicating portfolio.**

We define:

- x_1 = price of bond with risk-free interest rate r
- x_2 = number of stocks at price S (also called "hedge ratio" or "delta")

We assume:

- continuous compounding of interest ($FV = e^{ru} PV$)
- $d < e^{r} < u$

$\Rightarrow$ cost of replicating portfolio is $C = x_1 + S x_2$

(value at expiration: • if stock goes up: $e^r x_1 + S u x_2$

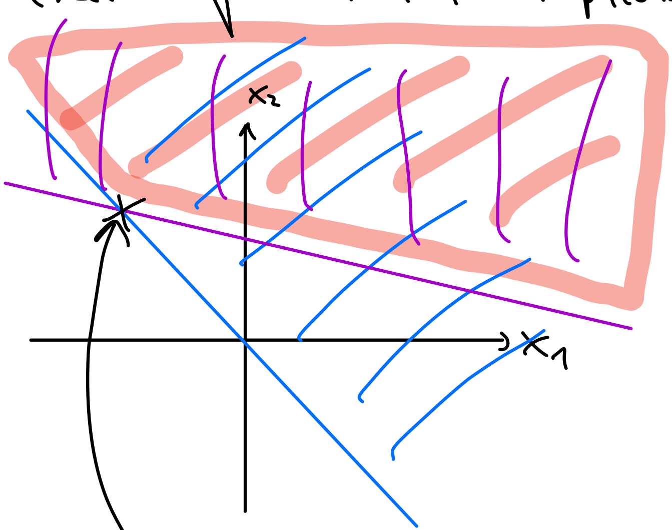
• if stock goes down: $e^r x_1 + S d x_2$)

(Note: for simplicity we here assume e^r is the discount factor for the one period under consideration (one step).)

In general: riskless or zero profit for seller if

- $e^r x_1 + S u x_2 \geq C_u$, where C_u = payoff in "up" scenario
- $e^r x_1 + S d x_2 \geq C_d$, where C_d = payoff in "down" scenario

(recall in general that for call options $C_u = \max(0, Su - K)$, $C_d = \max(0, Sd - K)$.)



in this region seller would always make profit

no risk-free profit, obligation is exactly met; here we set up the replicating portfolio $\Rightarrow$ option price

Option price is $C = x_1 + Sx_2$ with x_1 and x_2 determined by:

- $e^r x_1 + Su x_2 = C_u$
- $e^r x_1 + Sd x_2 = C_d$

Note: in our previous example we have
- $x_1 + 4000 x_2 = 1000$
- $x_1 + 2000 x_2 = 0$

$\Rightarrow 2000 x_2 = 1000 \Rightarrow x_2 = \frac{1}{2}, x_1 = -1000$

$\Rightarrow$ option price: $C = x_1 + Sx_2 = -1000\$ + 2500\$ \frac{1}{2} = 250\$$

	seller: borrow 1000\$, get 250\$ for option, buy $\frac{1}{2}$ stock for 1250\$.	buyer: buy option for 250\$
$S(T) = 4000\$$ ("up" scenario)	• need to buy another $\frac{1}{2}$ stock for 2000\$ • obligation: sell 1 stock for 3000\$ to buyer. $\Rightarrow$ profit: $-1000\$ - 2000\$ + 3000\$ = 0\$$	• buy stock for 3000\$ $\Rightarrow$ profit: $\underbrace{1000\$}_{4000\$ - 3000\$} - 250\$ = 750\$$
$S(T) = 2000\$$ ("down" scenario)	• $\frac{1}{2}$ stock is now worth 1000\$. $\Rightarrow$ profit: $-1000\$ + 1000\$ = 0\$$	$\Rightarrow$ profit: $-250\$$

note: • one can buy $\frac{1}{2}$ of a stock; this is called "fractional share" (also, e.g., for dividend reinvestments)

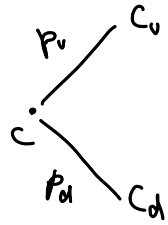
Generally, we need to solve
- $e^r x_1 + S u x_2 = C_u$
- $e^r x_1 + S d x_2 = C_d$

$\Rightarrow$ first eq. minus second: $\quad S u x_2 - S d x_2 = C_u - C_d \Rightarrow x_2 = \frac{1}{S}\frac{C_u - C_d}{u-d}$

To get x_1:
$$x_1 = e^{-r}\left(C_d - S d x_2\right)$$
$$= e^{-r}\left(C_d - d\,\frac{(C_u - C_d)}{u-d}\right)$$
$$= e^{-r}\left(\frac{C_d(u-d) - d(C_u - C_d)}{u-d}\right)$$
$$= e^{-r}\left(\frac{u C_d - d C_u}{u-d}\right)$$

option price is $C = x_1 + S x_2$
$$= e^{-r}\left(\frac{u C_d - d C_u}{u-d}\right) + \frac{C_u - C_d}{u-d}$$
$$= e^{-r}\left(\boxed{\underbrace{\frac{u - e^r}{u-d}}_{=:p_d}} C_d + \boxed{\underbrace{\frac{e^r - d}{u-d}}_{=:p_u}} C_u\right)$$

$$\Rightarrow \boxed{C = e^{-r}\left(p_d C_d + p_u C_u\right)}$$



Note:
- $p_d + p_u = \frac{u - e^r + e^r - d}{u-d} = 1$
- since $d < e^r < u$, we have that $0 < p_d < 1$ and $0 < p_u < 1$

$\Rightarrow$ it makes sense to call p_u, p_d probabilities, they are called $\boxed{\text{risk-neutral probabilities.}}$

Why? Look at expectation value of stock at time T under these risk-neutral probabilities: $\mathbb{E}(S(T)_{p_u,p_d}) = p_u Su + p_d Sd$

$$= \frac{e^r-d}{u-d} Su + \frac{u-e^r}{u-d} Sd$$

$$= S\left(\frac{ue^r - du + ud - e^r d}{u-d}\right)$$

$$= Se^r$$ $\Rightarrow$ expected value is the same as for the risk-free bond market

- remarkable: result $C = e^{-r}\left(p_d C_d + p_u C_u\right)$ is independent of the probabilities from our stock model!

- $\boxed{\text{In words: option price = discounted expectation value of the payoff under the risk-neutral probabilities}}$