Stochastic Modeling and Financial Mathematics — Week 5

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Lecture notes from Fall 2025

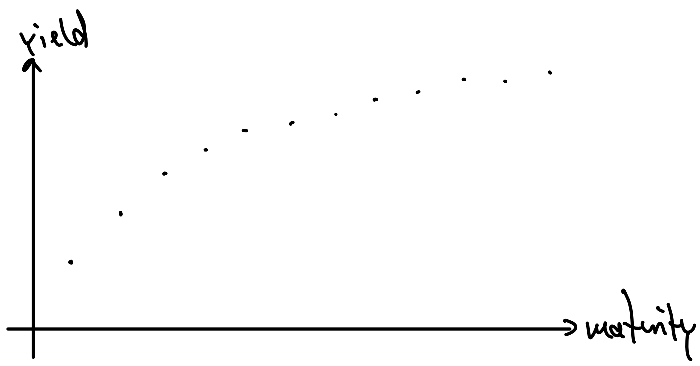
1.5 Spot Rates

General idea: yields / interest rates should be different for different maturities

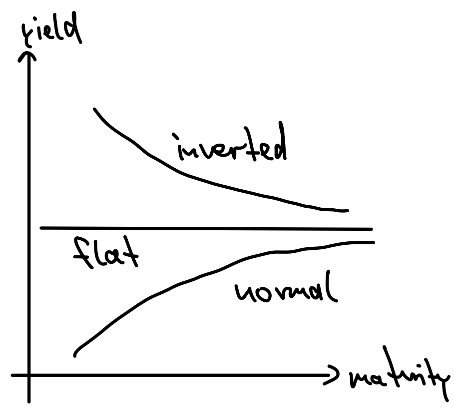
Usually: longer comitment (maturity) $\Rightarrow$ higher interest

This phenomenon is called "term structure".

A "normal" yield curve:



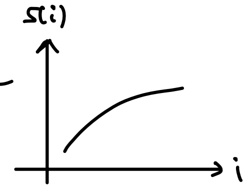
Other common types of curves:



Spot rate $S(i)$ = yield to maturity of i-period zero-coupon bond

(they follow the relation $P = \frac{F}{(1+S(i))^i}$)

$\Rightarrow$ **spot rate curve** is the zero-coupon bond yield curve



Suppose the $S(i)$ are given by some standard, say, in the US the US-treasury zero-coupon bonds, then a better level-coupon bond price formula would be

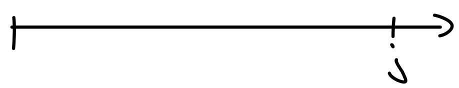
$$P = \sum_{i=1}^{n} \frac{C}{(1+S(i))^i} + \frac{F}{(1+S(n))^n} \qquad (m = 1 \text{ here})$$

Note: $d(i) := \frac{1}{(1+S(i))^i}$ are called **discount factors**.

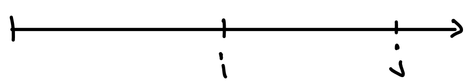
Note: risky bonds should be cheaper, this is often taken into account by adding a "static spread" s: $(1+S(i))^{-i} \longrightarrow (1+s+S(i))^{-i}$

Side remark: there is also the concept of **forward rates:**

consider 0-coupon bond



$$FV_j = P\,(1+S(j))^j$$



$$FV_i = P\,(1+S(i))^i$$

$$\Rightarrow FV_j = FV_i\,(1+S(i,j))^{j-i}$$
$$= P\,(1+S(i))^i\,(1+S(i,j))^{j-i}$$

$S(i,j) = (j-i)$-period spot rate i periods from now.

The $S(i,j)$ are in general unknown, and there are many stochastic models for them.

The interesting point here is that interest rate models have this 2-dimensional structure (i.e., S depends on two variables: i and j).

Based on the above, we can find the implied forward rates $f(i,j)$

$f(i,j)$ = model for $S(i,j)$ based on

$$(1+S(j))^{j} = (1+S(i))^{i}\,(1+f(i,j))^{j-i}$$

$$\Rightarrow f(i,j) = \left(\frac{(1+S(j))^{j}}{(1+S(i))^{i}}\right)^{\frac{1}{j-i}} - 1$$