Stochastic Modeling and Financial Mathematics Week 7

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Lecture notes from Fall 2025

Quiz 3 Problem 1:

$$\begin{array}{lll} & \nearrow 360 & \Longrightarrow u = \frac{360}{150} = \frac{12}{5} = 2.4 \\ S = 150 & & K = 260 \\ & \searrow 60 & \Longrightarrow d = \frac{60}{150} = \frac{2}{5} = 0.4 \end{array}$$

$$x_1 + 360 x_2 = \text{payoff(up)} = 360 - 260 = 100$$

$$x_1 + 60 x_2 = \text{payoff(down)} = 0$$

$$\Longrightarrow 300 x_2 = 100, \quad x_2 = \frac{1}{3} \Longrightarrow x_1 = -20$$

$$\Longrightarrow \text{price} = x_1 + \frac{1}{3} \cdot 150 = -20 + 50 = \underline{\underline{30}}$$

risk-neutral probabilities: $p = \frac{1-d}{u-d} = \frac{S_0 - S_d}{S_u - d_d} = \frac{150-60}{360-60} = \frac{90}{300} = \frac{3}{10}$

$$\text{price} = \frac{3}{10} \cdot 100 + \frac{7}{10} \cdot 0 = \underline{\underline{30}}$$

2.4 Binomial Tree and Calibration

We want to choose u, d, p_s in such a way that they match a given (observed) expectation value and variance, for large n (many steps).

We had $P(j,n) = \binom{n}{j} p_s^j (1-p_s)^{n-j}$, $S_T(j\ up) = S_0 u^j d^{n-j}$

Now let's put $S_T(j\ up) = S_0 e^{Y_j}$, $Y_j :=$ stock's rate of return

$$\Rightarrow \quad Y_j = \ln \frac{S_T(j\ up)}{S_0} = \ln u^j d^{n-j} = \ln \left(\frac{u}{d}\right)^j d^n \underset{\substack{\uparrow \\ \ln(ab) = \ln a + \ln b \\ \ln(a^x) = x \ln a}}{=} j \ln\left(\frac{u}{d}\right) + n \ln d$$

Next: compute expectation value and variance of Y_j (Y as a fct. of j).

Def.:

- Expectation value of X is $\mathbb{E}(X) = \sum_{j=0}^{n} X_j P(j,n)$
- Variance of X is $\mathrm{Var}(X) = \mathbb{E}\left(X - \mathbb{E}(X)\right)^2$

A few computational rules:

- $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$, $\mathbb{E}(\lambda X) = \lambda \mathbb{E}(X)$
- $\mathrm{Var}(X) = \mathbb{E}\left(X^2 - 2X\mathbb{E}(X) + \mathbb{E}(X)^2\right) = \mathbb{E}(X^2) - 2\mathbb{E}(X)\mathbb{E}(X) + \mathbb{E}(X)^2$

$$= \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

- $\mathrm{Var}(\lambda x) = \lambda^2\,\mathrm{Var}(x)$

- $\mathrm{Var}(x+y) = \mathbb{E}((x+y)^2) - \left(\mathbb{E}(x+y)\right)^2$

$$= \underline{\mathbb{E}(x^2)} + 2\,\mathbb{E}(xy) + \underline{\mathbb{E}(y^2)} - \left(\underline{\mathbb{E}(x)^2} + 2\,\mathbb{E}(x)\mathbb{E}(y) + \underline{\mathbb{E}(y)^2}\right)$$

$$= \underline{\mathrm{Var}(x)} + \underline{\mathrm{Var}(y)} + 2\underbrace{\left(\mathbb{E}(xy) - \mathbb{E}(x)\mathbb{E}(y)\right)}$$

$= \mathrm{Cov}(x,y) =$ covariance; does not generally vanish, but is zero for independent x and y

Next: y_j is a linear fct. of j, so we need to compute $\mathbb{E}$ and Var of $x_j = j$ (the identity fct.), i.e., $\underbrace{\mathbb{E}(x) \equiv \mathbb{E}(x_j) \equiv \mathbb{E}(j) \equiv \mathbb{E}(\mathbb{1}) \equiv \mathbb{E}(id)}_{\text{different notations for same object}}$.

$$\mathbb{E}(x) = \sum_{j=0}^{n} j\,P(j,n) = \sum_{j=0}^{n} j \binom{n}{j} p_s^j (1-p_s)^{n-j}$$

$$= (1-p_s)^n \sum_{j=0}^{n} j \binom{n}{j} \left(\frac{p_s}{1-p_s}\right)^j$$

note: $\sum_{j=0}^{n} j\binom{n}{j} z^j$ can be computed by shifting summation indices or via derivatives:

$$z\frac{d}{dz}\underbrace{\sum_{j=0}^{n}\binom{n}{j}z^j}_{=(1+z)^n} = z\sum_{j=0}^{n} j\binom{n}{j}z^{j-1} = \sum_{j=0}^{n} j\binom{n}{j}z^j$$

$$\Longrightarrow \sum_{j=0}^{n} j\binom{n}{j}z^j = z\frac{d}{dz}(1+z)^n = zn(1+z)^{n-1}$$

$$\Longrightarrow \mathbb{E}(x) = (1-p_s)^n \left(\frac{p_s}{1-p_s}\right) n \underbrace{\left(1+\frac{p_s}{1-p_s}\right)^{n-1}}_{\left(\frac{1}{1-p_s}\right)^{n-1}} \qquad \left(z = \frac{p_s}{1-p_s}\right)$$

$$= \left(\frac{p_s}{1-p_s}\right) n (1-p_s)$$

$$= n p_s$$

by similar computation: $\mathbb{E}(x^2) = n p_s \left[(n-1) p_s + 1 \right]$

$$\Longrightarrow Var(x) = \mathbb{E}(x^2) - \mathbb{E}(x)^2 = n p_s - n p_s^2 = n p_s (1-p_s)$$

Now we can compute $\mathbb{E}$ and Var of $Y_j = j \ln\left(\frac{v}{d}\right) + n \ln d$

We find:

- $\mathbb{E}(Y) = \mathbb{E}\left(j \ln\left(\frac{v}{d}\right) + n \ln d\right) = \ln\frac{v}{d}\, \mathbb{E}(j) + n \ln d = \left(\ln\frac{v}{d}\right) n p_s + n \ln d$

- $Var(Y) = Var\left(j \ln\frac{v}{d} + n \ln d\right) \underset{\substack{\uparrow \\ Cov(const, x) = 0 \\ Var(const) = 0}}{=} \left(\ln\frac{v}{d}\right)^2 Var(j) = \left(\ln\frac{v}{d}\right)^2 n p_s (1-p_s)$

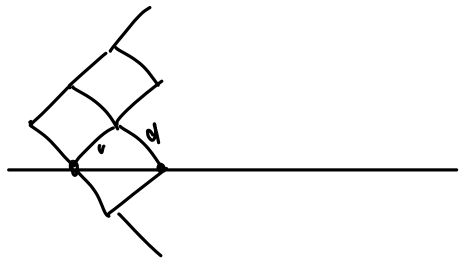
Next, we want to match $\mathbb{E}(Y)$ and $Var(Y)$ to given values:

$\mathbb{E}(Y_j) \xrightarrow{n \to \infty} \mu T$, $Var(Y_j) \xrightarrow{n \to \infty} \sigma^2 T$

μ is called mean-value

σ is called volatility

There is one more sensible condition: $ud = 1$



Same value for stock price if $ud=1$ on all horizontal lines

Then
- $\mathbb{E}(Y) = \left(\ln \frac{u}{d}\right) n p_s + n \ln d = 2(\ln u) n p_s - n \ln u = n(\ln u)(2p_s - 1)$
- $\operatorname{Var}(Y) = \left(\ln \frac{u}{d}\right)^2 n p_s (1-p_s) = 4(\ln u)^2 n p_s (1-p_s)$

Still there are several possible choices for u and p_s, a common one is

$$\boxed{p_s = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{T}{n}}, \quad u = e^{\sigma\sqrt{\frac{T}{n}}}} \qquad \left(d = e^{-\sigma\sqrt{\frac{T}{n}}}\right)$$

Check that they indeed give the right $\mathbb{E}$ and Var for large n:

- $\mathbb{E}(Y_j) = n\sigma\sqrt{\frac{T}{n}}\,\frac{\mu}{\sigma}\sqrt{\frac{T}{n}} = \mu T$
- $\operatorname{Var}(Y_j) = 4\left(\sigma\sqrt{\frac{T}{n}}\right)^2 n \left(\frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{T}{n}}\right)\left(\frac{1}{2} - \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{T}{n}}\right)$

$$= \sigma^2 T \left(1 + \frac{\mu}{\sigma}\sqrt{\frac{T}{n}}\right)\left(1 - \frac{\mu}{\sigma}\sqrt{\frac{T}{n}}\right)$$

$$= \sigma^2 T - \underbrace{\sigma^2 T\left(\frac{\mu}{\sigma}\sqrt{\frac{T}{n}}\right)^2}_{\to 0 \text{ as } n\to\infty} \xrightarrow{n\to\infty} \sigma^2 T$$

Note again: μ and σ can be read off from real data (we will talk about this later), therefore we want to choose $u, d, \underbrace{p_s}_{\text{not needed for option pricing}}$ depending on μ, σ.

Note: For the choice above, u does not depend on μ, only on σ

$\Rightarrow$ our option price is independent of μ!

2.5 Central Limit Theorem

For the limit $n \to \infty$ in our binomial tree model, we need to consider the limit $n \to \infty$ for the binomial distribution first:

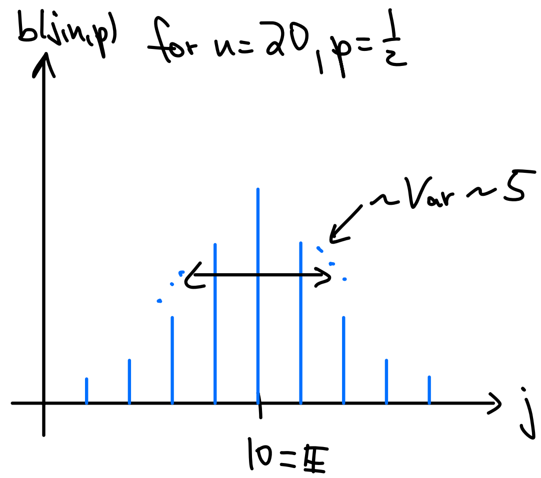
$\nearrow$ "up" with probability p

$\searrow$ "down" with probability $1-p$

Probability for j "up"s is $b(\underset{\text{number of "up"s}}{j}; \overset{\text{total number of steps}}{n}, \overset{\text{probability for "up"}}{p}) = \binom{n}{j} p^j (1-p)^{n-j}$ $\quad \left(\binom{n}{j} = \frac{n!}{(n-j)!\,j!} \right)$

Recall: • $\mathbb{E}(j) = np$

• $\mathrm{Var}(j) = np(1-p)$



Note: in order to compare distributions (here, pictures for different n), we need to center and normalize the variance

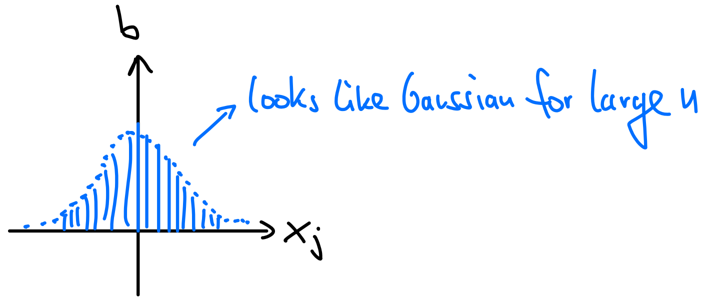
- centering: introduce $y_j = j - \mathbb{E}(j) = j - np$, such that

$$\mathbb{E}(y_j) = \mathbb{E}(j - np) = \mathbb{E}(j) - np = 0$$

- normalize variance (plus centering): $x_j = \dfrac{j - np}{\sqrt{np(1-p)}}$

$$\Rightarrow \underset{\uparrow \atop Var(\lambda X) = \lambda^2 Var(X)}{Var(x_j)} = \frac{1}{np(1-p)} \overbrace{Var(j - np)}^{= Var(j) = np(1-p)} = 1$$

$$\Rightarrow \quad j = \sqrt{np(1-p)}\, x_j + np$$



Next: look at cumulative distribution, meaning the probability for A or fewer "up"s.

It is given by $\displaystyle\sum_{j=0}^{A} b(j; n, p)\, \underbrace{\Delta j}_{=1 = \text{distance between } j\text{'s}}$

With the change of variables above, $j = \sqrt{np(1-p)}\, x_j + np$ and so $\Delta j = \sqrt{np(1-p)}\, \Delta x_j$,

so we should get (let A also depend on n)

$$\sum_{j=0}^{A_n} b(j; n, p)\, \Delta j = \sum_{x = \frac{-np}{\sqrt{np(1-p)}}}^{\frac{A_n - np}{\sqrt{np(1-p)}}} b\left(\sqrt{np(1-p)}\, x + np, n, p\right) \sqrt{np(1-p)}\, \Delta x$$

$$\xrightarrow[\text{should}]{n \to \infty} \int_{-\infty}^{\tilde{A}} \varphi(x)\, dx$$

$\tilde{A} \leftarrow$ if A_n is chosen nicely (e.g., $A_n = np + \tilde{A}\sqrt{np(1-p)}$)

φ: some limiting fct.

Such a convergence result is called $\boxed{\text{Central Limit Theorem (CLT)}}$.

For the binomial distribution, we get:

$$\boxed{\begin{array}{l} \sqrt{np(1-p)}\, b\left(\sqrt{np(1-p)}\, x + np, n, p\right) \xrightarrow{n\to\infty} \varphi(x) \quad \text{pointwise} \\[2ex] \text{with} \quad \varphi(x) = \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}_{\text{normalized Gaussian}} = \underbrace{\mathcal{N}}_{\text{"normal distribution"}}(\overset{\text{mean}}{0}, \underset{\text{variance}}{1}) \end{array}}$$

Remarks:

- Here we get pointwise convergence, but generally the CLT gives us convergence in the sense of cumulative distribution functions.
- Let's check normalization: $\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \right)^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; e^{-\left(\frac{x^2+y^2}{2}\right)}$

polar coordinates

$x^2+y^2=r^2$

$dxdy = rdrd\varphi$

$$= \frac{1}{2\pi} \int_0^{\infty} dr \int_0^{2\pi} d\varphi \; r e^{-\frac{r^2}{2}}$$

$$= \int_0^{\infty} dr \; r e^{-\frac{r^2}{2}}$$

$$= -e^{-\frac{r^2}{2}} \Big|_0^{\infty}$$

$$= 1$$

- One can also check that indeed $\mathbb{E}(x) = 0$ and $\mathrm{Var}(x) = 1$

Ingredients for the proof:

- We need to approximate factorials with the $\boxed{\text{Stirling approximation } n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n}$

Motivation why this is true: $\ln n! = \sum_{i=1}^{n} \ln(i) \approx \int_1^n \ln(x) dx = \int_1^n 1 \cdot \ln(x) dx$

$$= x \ln x \Big|_1^n - \int_1^n x \frac{1}{x} dx = n\ln(n) - (n-1) \approx n \ln(n) - n$$

$$\Rightarrow n! = e^{\ln n!} \approx e^{n\ln(n)-n} \overset{\ln a^b = b\ln a}{=} e^{\ln n^n} e^{-n} = n^n e^{-n} = \left(\frac{n}{e}\right)^n$$

(the factor $\sqrt{2\pi n}$ (or even higher order terms) can be found with more careful arguments)

- Taylor expansion

2.6 Black-Scholes Formula

Recall the exact formula for the price of European call options:

$$C = e^{-rT} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \max(0, S u^j d^{n-j} - K) \quad (K \text{ strike price}, r \text{ period interest rate})$$

$$= e^{-rT}\, \underbrace{\mathbb{E}(\text{payoff})}_{\text{under binomial distribution}}$$

Many terms in the sum are 0; payoff $\neq 0$ if $Su^j d^{n-j} - K > 0$

$$\Rightarrow \text{need } u^j d^{n-j} > \frac{K}{S} \Rightarrow \left(\frac{u}{d}\right)^j > \frac{K}{Sd^n} \Rightarrow j > \frac{\ln\left(\frac{K}{Sd^n}\right)}{\ln\left(\frac{u}{d}\right)} = \frac{\ln\left(\frac{K}{S}\right) - n\ln d}{\ln\left(\frac{u}{d}\right)} =: A_n$$

$$\Rightarrow C = e^{-rT} \sum_{j=A_n}^{n} \binom{n}{j} p^j (1-p)^{n-j} \left(S u^j d^{n-j} - K\right)$$

$$= S \sum_{j=A_n}^{n} \binom{n}{j} \left(pue^{-r\frac{T}{n}}\right)^j \left((1-p)de^{-r\frac{T}{n}}\right)^{n-j} - Ke^{-rT} \sum_{j=A_n}^{n} \binom{n}{j} p^j (1-p)^{n-j}$$

Recall $p = \dfrac{e^{r\frac{T}{n}} - d}{u-d} \Rightarrow 1-p = \dfrac{u-d-e^{r\frac{T}{n}}+d}{u-d} = \dfrac{u-e^{r\frac{T}{n}}}{u-d}$

$$\Rightarrow (1-p)\,d e^{-r\frac{T}{n}} = \frac{u-e^{r\frac{T}{n}}}{u-d}\,d e^{-r\frac{T}{n}} = \frac{ude^{-r\frac{T}{n}}-d+u-u}{u-d} = 1-pue^{-r\frac{T}{n}}$$

$$\Rightarrow C = S\sum_{j=A_n}^{n} b\left(j,n,pue^{-r\frac{T}{n}}\right) - Ke^{-r\frac{T}{n}}\sum_{j=A_n}^{n} b(j,n,p)$$

Next: use our calibration $u = e^{\sigma\sqrt{\frac{T}{n}}}$, $d = \frac{1}{u} = e^{-\sigma\sqrt{\frac{T}{n}}}$

$$\Rightarrow A_n = \frac{\ln\left(\frac{K}{S}\right) - n\ln d}{\ln\left(\frac{u}{d}\right)} = \frac{\ln\left(\frac{K}{S}\right) - n\ln e^{-\sigma\sqrt{\frac{T}{n}}}}{\ln e^{2\sigma\sqrt{\frac{T}{n}}}} = \frac{\ln\left(\frac{K}{S}\right)+\sigma\sqrt{T}\sqrt{n}}{2\sigma\sqrt{\frac{T}{n}}} = \frac{\ln\left(\frac{K}{S}\right)}{2\sigma\sqrt{T}}\sqrt{n} + \frac{1}{2}n$$

$$\Rightarrow p = \frac{e^{r\frac{T}{n}}-d}{u-d} = \frac{e^{r\frac{T}{n}}-e^{-\sigma\sqrt{\frac{T}{n}}}}{e^{\sigma\sqrt{\frac{T}{n}}}-e^{-\sigma\sqrt{\frac{T}{n}}}} \overset{\text{Taylor}}{=} \frac{1+r\frac{T}{n}+\overbrace{O(n^{-2})}^{\text{"big O" notation}} - \left[1-\sigma\sqrt{\frac{T}{n}}+\frac{1}{2}\sigma^2\frac{T}{n}+O(n^{-\frac{3}{2}})\right]}{1+\sigma\sqrt{\frac{T}{n}}+\frac{1}{2}\sigma^2\frac{T}{n}+O(n^{-\frac{3}{2}}) - \left[1-\sigma\sqrt{\frac{T}{n}}+\frac{1}{2}\sigma^2\frac{T}{n}+O(n^{-\frac{3}{2}})\right]}$$

$$= \frac{\sigma\sqrt{\frac{T}{n}} + \left(r-\frac{\sigma^2}{2}\right)\frac{T}{n} + O(n^{-\frac{3}{2}})}{2\sigma\sqrt{\frac{T}{n}} + O(n^{-\frac{3}{2}})}$$

$$= \frac{1}{2}\left(1 + \frac{\left(r-\frac{\sigma^2}{2}\right)}{\sigma}\sqrt{\frac{T}{n}} + \underbrace{O(n^{-1})}_{=\frac{O(n^{-\frac{3}{2}})}{2\sigma\sqrt{T}n^{-\frac{1}{2}}}}\right)$$

Then the integral boundary from the CLT is

$$\hat{A} = \lim_{n\to\infty}\frac{A_n-np}{\sqrt{np(1-p)}} = \lim_{n\to\infty}\frac{\frac{\ln\left(\frac{K}{S}\right)}{2\sigma\sqrt{T}}\sqrt{n}+\frac{1}{2}n-\left[\frac{n}{2}+\frac{n}{2}\frac{\left(r-\frac{\sigma^2}{2}\right)}{\sigma}\sqrt{\frac{T}{n}}+O(n^{-1})\right]}{\underbrace{\sqrt{n}\left(\frac{1}{2}+\frac{\text{const}}{\sqrt{n}}+O(n^{-1})\right)^{\frac{1}{2}}\left(\frac{1}{2}-\frac{\text{const}}{\sqrt{n}}+O(n^{-1})\right)^{\frac{1}{2}}}_{=\sqrt{n}\sqrt{\frac{1}{4}+O(n^{-1})}=\sqrt{n}\frac{1}{2}+O(n^{-\frac{1}{2}})}}$$

$$= \frac{\ln\left(\frac{K}{S}\right)-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

Now applying the CLT gives us: (after some more computation)

$$\boxed{C = S\,\Phi(x) - Ke^{-rT}\,\Phi(x - \sigma\sqrt{T})\text{, the Black-Scholes formula,}}$$

with cumulative normal distribution $\Phi(x) = \int_{-\infty}^{x} \underbrace{\varphi(y)}_{\text{normal Gaussian}}\,dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy$

and $x = \dfrac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$.

let's check: $\hat{A} = \dfrac{\ln\left(\frac{K}{S}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = -\left(\dfrac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$

$$= -\left(x - \sigma\sqrt{T}\right)$$

Then: $Ke^{-rT} \sum_{j=A_n}^{n} b(j,n,p) \xrightarrow[\text{CLT}]{n\to\infty} Ke^{-rT} \int_{\hat{A}}^{\infty} \varphi(y)\,dy = Ke^{-rT} \int_{-(x-\sigma\sqrt{T})}^{\infty} \varphi(y)\,dy$

$$= Ke^{-rT} \int_{-\infty}^{x-\sigma\sqrt{T}} \varphi(y)\,dy$$

$$= Ke^{-rT}\,\Phi(x - \sigma\sqrt{T})$$

Similar for the first summand, but we need to use approximations for $pue^{-r\frac{T}{n}}$.