Stochastic Modeling and Financial Mathematics — Week 7

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Lecture notes from Fall 2025

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## Summary of last few sessions

Binary model for stock price:

$$\begin{tikzcd} & Su \\ S \arrow[ur] \arrow[dr] & \\ & Sd \end{tikzcd}$$

Binary model for option price:

$$\begin{tikzcd} & C_u = \text{payoff if stock goes up} \\ C \arrow[ur] \arrow[dr] & \\ & C_d = \text{payoff if stock goes down} \end{tikzcd}$$

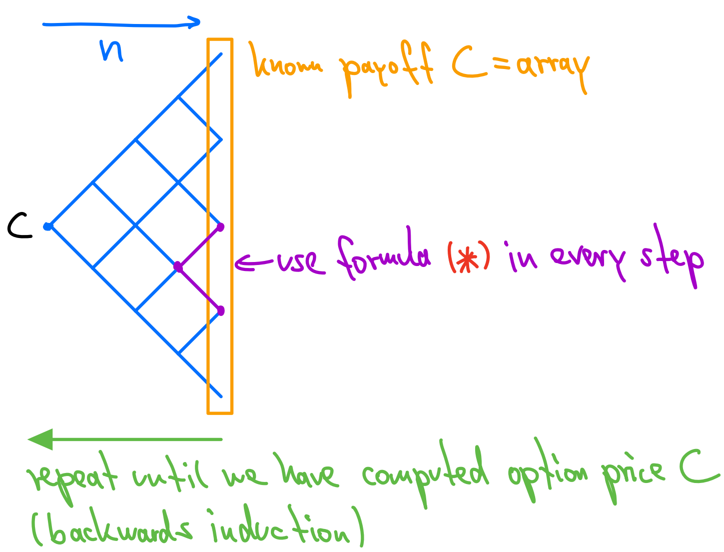
Option price = value of replicating portfolio $= C = \underbrace{x_1}_{\text{value of bonds}} + S\overbrace{x_2}^{\text{number of stocks}}$, with $x_1, x_2$ determined by

- $e^r x_1 + Su x_2 = C_u$
- $e^r x_1 + Sd x_2 = C_d$

Solving this gives $\underbrace{C = e^{-r}\left(p_d C_d + p_u C_u\right)}_{(*)}$ with $p_d = \frac{u-e^r}{u-d}$, $p_u = \frac{e^r - d}{u-d}$ $(p_d + p_u = 1)$

"risk-neutral probabilities"

Repeating this yields a binomial tree model for option pricing:

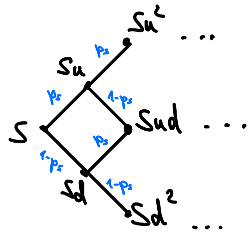


Note: binomial trees are very versatile and can be used, e.g., also for American options or with dividend payments. For the case of European call without dividends, an explicit formula is available (see also the discussion later today and in next session).

Question: How to choose parameters $u$ and $d$ ?

For historical stock data, one usually considers expectation and variance of the rate of return $y(t)$ from $S(t)=S_0 e^{y(t)}$, i.e., $y(t)=\ln \frac{S(t)}{S_0}$. They are denoted by

$\mathbb{E}(y(t)) = \mu t$ and $Var(y(t)) = \sigma^2 t$.

We want to match these to our tree model 

, where we find

$$\mathbb{E}(y(n\text{ steps})) = \left(\ln\frac{u}{d}\right) n p_s + n \ln d \quad \text{and} \quad Var(y(n\text{ steps})) = \left(\ln\frac{u}{d}\right)^2 n p_s(1-p_s).$$

Now the choice $p_s = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{t}{n}}$, $u = e^{\sigma\sqrt{\frac{t}{n}}}$, $d = \frac{1}{u}$ yields $\mathbb{E}(y(t))$ and $Var(y(t))$ from above in the limit $n \to \infty$.

Noticeable here is that $u$ and $d$, and thus $p_u$ and $p_d$, and thus the option price do <u>not</u> depend on the average growth of the stock $\mu$, but only on its volatility $\sigma$.

For European call options one can easily solve the recursion of the binomial tree:

$$C = e^{-rT} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \max(0, S u^j d^{n-j} - K)$$

Taking the limit $n \to \infty$, using the Central Limit Theorem, yields the Black-Scholes formula:

$$C_{n\to\infty} = S \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy - K e^{-rT} \int_{-\infty}^{x-\sigma\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy, \quad \text{with } x = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$