Quiz 4 Problem 1:

$$\mathbb{E}(Y) = \left(\ln \frac{u}{d}\right) np + n \ln d$$

$$\mathrm{Var}(Y) = \left(\ln \frac{u}{d}\right)^2 np(1-p)$$

Now: $p = \frac{1}{2}$, $u = e^{\mu \frac{T}{n} + \sigma \sqrt{\frac{T}{n}}}$, $d = e^{\mu \frac{T}{n} - \sigma \sqrt{\frac{T}{n}}}$

$$\Rightarrow \mathbb{E}(Y) = \underbrace{\left(\ln e^{2\sigma \sqrt{\frac{T}{n}}}\right)}_{2\sigma\sqrt{\frac{T}{n}}} n \frac{1}{2} + n\left(\mu \frac{T}{n} - \sigma \sqrt{\frac{T}{n}}\right) = \sigma \sqrt{n}\sqrt{T} + \mu T - \sigma\sqrt{n}\sqrt{T} = \mu T$$

$$\mathrm{Var}(Y) = \left(2\sigma \sqrt{\frac{T}{n}}\right)^2 n \frac{1}{2} \frac{1}{2} = \sigma^2 T$$

2.8 Monte-Carlo Method

Idea: use random samplings to approximate expectation values

Ex.: binomial tree model for option price of European calls:

$$C = \sum_{j=0}^{n} b(j,n,p) \underbrace{e^{-rT} \max(S u^j d^{n-j} - K, 0)}_{= f(j,n)} = \mathbb{E}_{\text{bin.}}(f)$$

Monte-Carlo: take m samples $j_1, \dots, j_m$ from bin. distribution ($m \ll n$) and compute

$\frac{1}{m} \sum_{k=1}^{m} f(j_k, n)$, the empirical mean, or sample average.

The law of large numbers says that $\boxed{\frac{1}{m} \sum_{k=1}^{m} f(j_k, n) \xrightarrow{m \to \infty} \underbrace{\mathbb{E}(f)}_{\text{theoretical expectation}}.}$

(for very general classes of probability distributions)

Idea/hope of Monte-Carlo method:

- time efficient / fast, since only $m \ll n$ steps are necessary to compute a good approximation
- interesting idea: use randomness to approximate a deterministic quantity

3. Continuous Time Models

3.1 Brownian Motion

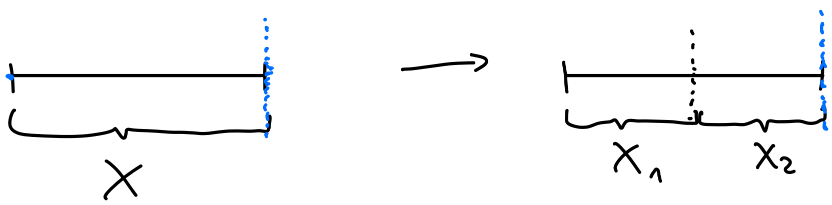
Motivation: Let us consider the normal distribution with mean 0 and variance 1:

$$\mathcal{N}(\underbrace{0}_{\text{mean}},\underbrace{1}_{\text{standard deviation}}) \qquad (\mathcal{N}(\mu,\sigma) \text{ for mean } \mu, \text{ std. deviation } \sigma)$$

Importance of the normal distribution comes from the Central Limit Theorem (CLT)!
(Which holds under very weak conditions on the underlying probability distribution; in particular, random variables need to be independent or only weakly dependent.)

Now consider a random variable X that is normally distributed: $X \underset{\text{is distributed according to}}{\sim} \mathcal{N}(0,1)$

Consider splitting it into two: $X = X_1 + X_2$, s.t. X_1 and X_2 independent and same distribution.



Note: $0 = \mathbb{E}(X) = \mathbb{E}(X_1+X_2) = \mathbb{E}(X_1)+\mathbb{E}(X_2) \overset{X_1 \text{ and } X_2 \text{ same distribution}}{=} 2\,\mathbb{E}(X_1) \Rightarrow X_1$ has expectation 0

How does the variance behave?

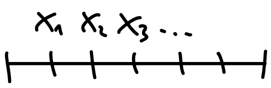
$$1 = Var(X) = Var(X_1+X_2) \overset{X_1 \text{ and } X_2 \text{ independent}}{=} Var(X_1)+Var(X_2) \overset{\text{same dist.}}{=} 2\,Var(X_1) \Rightarrow Var(X_1) = \frac{1}{2}$$

Computing higher moments shows that $X_1 + X_2$ is normally distributed.

$\Rightarrow X_1$ is distributed according to $\underbrace{\mathcal{N}\left(0, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\mathcal{N}(0,1)}$ $\swarrow$ $Var\left(\frac{1}{\sqrt{2}}X\right) = \frac{1}{2}Var(X)$

both are normal distributions with mean 0 and variance $\frac{1}{2}$, so they are the same

for n steps:



$$1 = Var(X) = Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) = n\,Var(X_1)$$

$$\Rightarrow Var(X_1) = \frac{1}{n}$$

$$\Rightarrow \text{each } X_i \sim \mathcal{N}\left(0, \frac{1}{\sqrt{n}}\right) = \frac{1}{\sqrt{n}}\mathcal{N}(0,1)$$

or, calling $\frac{1}{n} = \Delta t$: $X_i \sim \sqrt{\Delta t}\,\mathcal{N}(0,1)$

This motivates the following definition:

$\hookrightarrow$ for fixed t, $W(t)$ is a random variable

<u>Def.:</u> A stochastic process $t \longmapsto W(t)$ ($\swarrow \mathbb{R}$) for $t \geq 0$ is called $\boxed{\text{Brownian Motion (BM)}}$ or $\boxed{\text{Wiener process}}$ if:

a) $W(0) = 0$ (convention)

b) each realization is continuous,

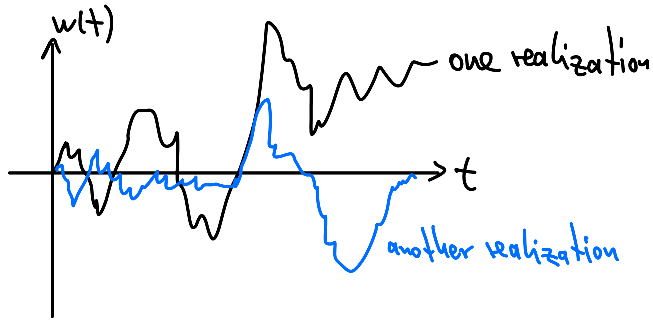
c) for any $0 \leq s_1 < s_2 < t_1 < t_2$ the increments

$W(t_2) - W(t_1)$ and $W(s_2) - W(s_1)$ are independent,

d) $W(t_2) - W(t_1)$ is distributed according to $\sqrt{t_2 - t_1}\,\mathcal{N}(0,1)$ for all $0 \leq t_1 < t_2$.

Note: one can indeed show that such a process exists and is unique

More pictures in python next time:



Is that a good model for stock prices?

No: • BM can become negative

(• parameters similar to μ and σ in calibrated binomial tree are missing)

Solution (better stock price model):

We use Geometric Brownian Motion (GBM): $\boxed{S(t) = S(0)\, e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}}$.

It turns out (see next HW sheet) that the calibrated binomial tree model converges for $n \to \infty$ indeed to GBM!

A few notes on python implementation of BM:

• BM: $W_0 = 0$

$W_1 = \sqrt{\Delta t} \cdot$ sample from $\mathcal{N}(0,1)$

$W_2 = W_1 + \sqrt{\Delta t} \cdot$ sample from $\mathcal{N}(0,1)$

...

in python: $dW = \text{normal}(0, 1, \text{size}=n) \cdot \sqrt{\Delta t}$ $\longrightarrow$ vector

$W = \text{cumsum}(dW)$ (= vector containing entries of cumulative sum)

$$= \begin{pmatrix} dW[0] \\ dW[0]+dW[1] \\ \vdots \\ dW[0]+\dots+dW[n-1] \end{pmatrix}$$

$$W = \underbrace{r\_[0, W]}$$ (add 0 at time 0)

r_[a,b] appends row vector b to vector a

$(a=(a_0,a_1,\dots,a_K)$, $b=(b_0,b_1,\dots,b_\ell)$ $\Rightarrow$ r_[a,b] $=(a_0,a_1,\dots,a_K,b_0,b_1,\dots,b_\ell))$

- ensemble of BMs:

$$dW = \text{normal}(0, 1, \text{size}=(M, N)) \cdot \sqrt{\Delta t}$$

#of time steps

#of samples

$$W = \text{cumsum}(dW, \text{axis}=1)$$

cumulative sum over row entries (axis=0 would be columns)

add zero vector

Note: • similarly one can use, e.g., mean(W, axis=0)

mean over samples

- W[:10:,:] selects 10 sample paths (plot(t, W[:10:,:].T))

transpose to plot rows

- seed(k) (k some number) gives you the same samples (same random numbers)

3.2 Stochastic Integrals

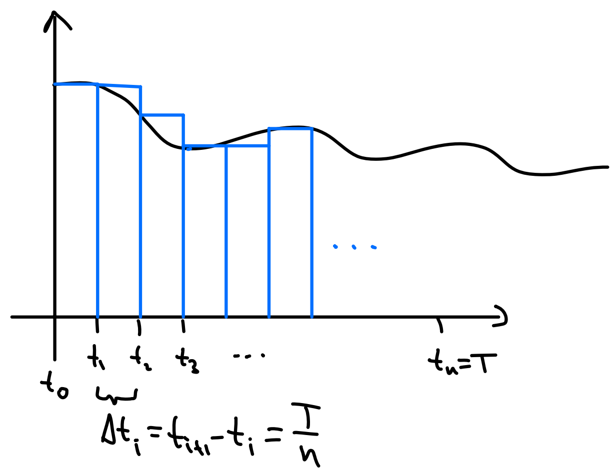
Later: want $\underbrace{\text{PDE}}_{\text{partial differential equation}}$ with randomness = stochastic partial differential equation = SPDE

to model stock market: $\quad dX = f\,dt + g\,dW \quad$, W Brownian motion

In order to make sense of such an equation, we need to know how to define $\int g\,dW$ (bc. $\frac{dW}{dt}$ does not make sense: BM is not differentiable)

Recall Riemann sum for Riemann integral:

$$\int_0^T f(t)\,dt = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\,\Delta t_i$$



There are two kinds of stochastic integrals: Itô and Stratonovich

$\boxed{\text{Itô-integral:}}$

Defined analogously to Riemann sum:

$$\int_0^T f(t)\,dW(t) := \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\,\Delta W_i \quad \text{with} \quad \overbrace{\Delta W_i}^{\text{random variable}} = W(t_{i+1}) - W(t_i) \overset{\text{distributed like}}{\sim} \sqrt{t_{i+1} - t_i}\,\mathcal{N}(0,1)$$

$$= \sqrt{\Delta t}\,\mathcal{N}(0,1)$$

evaluate f at beginning of interval

Stratonovich Integral:

evaluate f at middle of interval

$$\int_0^T f(t) \circ dW(t) = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i^*) \Delta W_i \qquad \text{with } t_i^* = \frac{t_{i+1}+t_i}{2}$$

notation to differentiate it from Itô integral

Note: One can show that this definition is equal to $\lim_{n\to\infty} \sum_{i=0}^{n-1} \left(\frac{f(t_{i+1})+f(t_i)}{2} \right) \Delta W_i$.

To illustrate their difference, consider the following example:

<u>Ex.:</u> integrate Brownian motion against itself: $\int_0^T W(t)\, dW(t) = \int_0^T W\, dW$

Note: If $W(t)$ were differentiable, we could use the chain rule $dW = \frac{dW}{dt} dt$

$$\Rightarrow \int_0^T W(t)\, dW(t) = \int_0^T W(t) \frac{dW(t)}{dt}\, dt = \frac{1}{2} \int_0^T \frac{d}{dt}\left(W(t)^2\right) dt = \frac{1}{2} W(T)^2 - \frac{1}{2} \underbrace{W(0)^2}_{=0}$$

But $W(t)$ is not differentiable!

Let us compute "by hand":

$$\int_0^T W(t)\, dW(t) := \lim_{n\to\infty} \sum_{i=0}^{n-1} W(t_i)\, \Delta W_i = \lim_{n\to\infty} \sum_{i=0}^{n-1} \underbrace{W(t_i)\left(W(t_{i+1}) - W(t_i)\right)}$$

$$= W(t_i)\, W(t_{i+1}) - W(t_i)^2$$

$$= \frac{1}{2}\left[W(t_{i+1})^2 - W(t_i)^2 - \underbrace{\left(W(t_{i+1}) - W(t_i)\right)^2}_{= W(t_{i+1})^2 - 2W(t_{i+1})W(t_i) + W(t_i)^2} \right]$$

$$\Rightarrow \int_0^T W(t)\,dW(t) = \frac{1}{2}\lim_{n\to\infty} \underbrace{\sum_{i=0}^{n-1}\left(W(t_{i+1})^2 - W(t_i)^2\right)}_{= W(T)^2 - W(0)^2 = W(T)^2 \quad \text{(telescope sum)}} - \frac{1}{2}\lim_{n\to\infty}\sum_{i=0}^{n-1} \Delta W_i^2$$

$$(W_1^2 - W_0^2) + (W_2^2 - W_1^2) + (W_3^2 - W_2^2) + \dots (W_n^2 - W_{n-1}^2) = W_n^2 - W_0^2$$

Question: How is $\sum_{i=0}^{n-1} \Delta W_i^2$ distributed?

We know that $\mathbb{E}(\Delta W_i^2) = \Delta t = \frac{T}{n}$ (recall that $\Delta W \sim \sqrt{\Delta t}\,\mathcal{N}(0,1)$)

$$\mathbb{E}(\Delta W_i^2) = \operatorname{Var}(\Delta W_i) = \mathbb{E}(\Delta W_i^2) - \underbrace{\mathbb{E}(\Delta W_i)^2}_{=0}$$

$$\Rightarrow \mathbb{E}\left(\sum_{i=0}^{n-1} \Delta W_i^2\right) = \sum_{i=0}^{n-1} \frac{T}{n} = T$$

What about variance?

$$\operatorname{Var}\left(\sum_{i=0}^{n-1} \Delta W_i^2\right) = \mathbb{E}\left(\sum_{i,j=0}^{n-1} \Delta W_i^2 \Delta W_j^2\right) - \underbrace{\mathbb{E}\left(\sum_{i=0}^{n-1} \Delta W_i^2\right)^2}_{T^2}$$

$$= \underbrace{\mathbb{E}\left(\sum_{i=0}^{n-1} \Delta W_i^4\right)}_{} + \underbrace{\mathbb{E}\left(\sum_{i\neq j} \Delta W_i^2 \Delta W_j^2\right)}_{} - T^2$$

One can compute: $\approx \sum_{i=0}^{n-1} \frac{T^2}{n^2} = \mathcal{O}\left(\frac{1}{n}\right)$

ΔW_i and ΔW_j independent!

$$= \sum_{i\neq j} \mathbb{E}(\Delta W_i^2 \Delta W_j^2) = n(n-1)\frac{T^2}{n^2} = T^2 + \mathcal{O}\left(\frac{1}{n}\right)$$

$$= \mathcal{O}\left(\frac{1}{n}\right)$$

$\Rightarrow$ Variance vanishes in the limit $n\to\infty$ $\Rightarrow \sum_{i=0}^{n-1} \Delta W_i^2 = T$, a constant

$\Rightarrow$ a deterministic process

$$\boxed{\text{Conclusion: } \int_0^T W(t)\,dW(t) = \frac{1}{2}W(T)^2 - \frac{1}{2}T}$$ (different from usual integral!)

let's consider same example as before with the Stratonovich integral:

$$\int_0^T W(t)\circ dW(t) = \lim_{n\to\infty}\sum_{i=0}^{n-1} \underbrace{W(t_i^*)\left(W(t_{i+1})-W(t_i)\right)}$$

$$\frac{1}{2}\left[W(t_{i+1})^2 - W(t_i)^2 + \left(W(t_i^*)-W(t_i)\right)^2 - \left(W(t_{i+1})-W(t_i^*)\right)^2\right]$$

Similar to before: $\cdot\ \mathbb{E}\left(\left(W(t_i^*)-W(t_i)\right)^2\right) = t_i^* - t_i = \frac{t_{i+1}+t_i}{2} - t_i = \frac{t_{i+1}-t_i}{2} = \frac{\Delta t}{2}$

$\cdot\ \mathbb{E}\left(\left(W(t_{i+1})-W(t_i^*)\right)^2\right) = t_{i+1} - \left(\frac{t_{i+1}+t_i}{2}\right) = \frac{\Delta t}{2}$

and variances vanishes

$$\Rightarrow \int_0^T W(t)\circ dW(t) = \frac{1}{2}W(T)^2 + \frac{T}{2} - \frac{T}{2} = \frac{1}{2}W(T)^2$$ (as we would expect from regular calculus)

In comparison:

- Stratonovich: $\cdot$ much "nicer", properties more similar to usual integration
  - $\cdot$ but in each step function is evaluated in between t_{i+1} and t_i
    ↳ undesirable for some SPDEs
- Itô: $\cdot$ technically a bit "harder" to handle, results different from regular calculus
  - $\cdot$ but, at each t_i, f is evaluated and an increment is added
    ↳ this is what we want for stock market SPDE later

Python hints for HW 6 :

- GBM: $S(t) = S(0)\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right)$

$$= S(0)\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\sum_{i=1}^{N}\Delta t + \sigma\sum_{i=1}^{N}\Delta W_i\right)$$

$$= S(0)\prod_{i=1}^{N}\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\Delta W_i\right)$$

$$= \underbrace{S(0)}_{=1 \text{ for Problem 2}}\ \text{cumprod}(\ldots)$$

- paths from binomial tree: hints: random_sample((M, N)) gives random sample from [0, 1] with uniform probability

(or look at "choice" fct.)

- Problem 3: only need $S(T)$ (need not generate full paths)

$$\left(S(T) = S_0\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W(T)\right)\right)$$

random variable $\sim \sqrt{T}\,\mathcal{N}(0,1)$

$(W(T) = W(T) - W(0))$

- Problem 4: $W = dW_0 + dW_1 + dW_2 + \ldots$

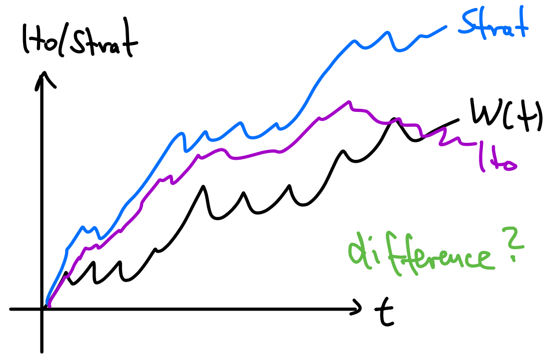
$$= (W_1 - W_0) + (W_2 - W_1) + (W_3 - W_2)$$

$$\Rightarrow \text{Ito} = W_0 \underbrace{(W_2 - W_0)}_{dW_0 + dW_1} + W_2 \underbrace{(W_4 - W_2)}_{dW_2 + dW_3} + \dots$$

$$\Rightarrow \text{Strat} = W_1 (W_2 - W_0) + W_3 (W_4 - W_2) + \dots$$

(recall W[a:b:inc] notation)

For exercise, just look at one realization:



Use cumsum to be able to plot $\int_0^t W(s)ds$ against t.