

3.3 Stochastic Differential Equations

Usual first-order ordinary differential equation (ODE):

$$\frac{dX(t)}{dt} = f(X(t), t)$$

We could also write this in integral form: $X(t) = X(0) + \int_0^t f(X(s), s) ds$

A stochastic differential equation (SDE) can be written down in integral form:

$$X(t) = X(0) + \int_0^t f(X(s), s) ds + \underbrace{\int_0^t g(X(s), s) dW(s)}_{\text{Brownian motion increments}}$$

Stochastic integral, I'll from now on

As a shorthand notation, we often write: $dX(t) = f(X(t), t) dt + g(X(t), t) dW(t)$

Today: Examples, numerical solutions and their error; next time: how to find solutions

Ex.: Next time, we will see that **GBM** $S(t) = S_0 \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W(t)\right)$

satisfies the SDE $dS(t) = \mu S(t) dt + \sigma S(t) dW(t)$

In integral form: $S(t) - S_0 = \mu \int_0^t S(u) du + \sigma \int_0^t S(u) dW(u)$

Even without knowing the solution, we can figure out its expectation value:

$$\mathbb{E}(S(t)) - S_0 = \mu \int_0^t \mathbb{E}(S(u)) du + \sigma \int_0^t \underbrace{\mathbb{E}(S(u) dW(u))}_{=0}$$

$$= \mathbb{E}(S(u)) \underbrace{\mathbb{E}(dW(u))}_{=0} = 0$$

bc. $dW(u)$ and $S(u)$ are independent (4th integral!)

$$\Rightarrow \mathbb{E}(S(t)) - S_0 = \mu \int_0^t \mathbb{E}(S(u)) du \quad \left(\frac{d\mathbb{E}(S(t))}{dt} = \mu \mathbb{E}(S(t)) \right)$$

$$\Rightarrow \text{solution: } \mathbb{E}(S(t)) = S_0 e^{\mu t}$$

Next: numerical solutions

Usual ODEs $\frac{dX(t)}{dt} = f(X(t), t)$ can be solved numerically with Euler's method:

partition $[0, t]$ into N steps, $x_n = n \Delta t$, $\Delta t = \frac{T}{N}$ and then write down discretized eq.:

$$\frac{x_{n+1} - x_n}{\Delta t} = f(x_n, t_n) \Rightarrow x_{n+1} = x_n + f(x_n, t_n) \Delta t$$

What is the error?

In one step: Euler: $x_1 = x_0 + f(x_0, 0) \Delta t$ $= f(x(0), 0)$

• Taylor expansion of exact solution $x(\Delta t) = x(0) + \overbrace{x'(0)}^{t_0} \Delta t + \frac{1}{2} x''(0) (\Delta t)^2 + \mathcal{O}(\Delta t^3)$

$$\Rightarrow |x(\Delta t) - x_1| \approx \text{const} (\Delta t)^2 \quad (\Delta t = \frac{T}{N})$$

$$\Rightarrow \text{total error } |x(t) - x_N| \approx \sum_{i=1}^N \text{const} (\Delta t)^2 \approx \frac{c(t)}{N}$$

The same works for SDEs; it is then called Euler-Maruyama method:

$$X_{n+1} = X_n + f(X_n, t_n) \Delta t + g(X_n, t_n) \Delta W_n$$

For the error, there are two often used definitions:

• strong error: $\mathbb{E}(|X(t) - X_n|) \approx c_s (\Delta t)^\alpha$, α = strong order of convergence

The relevance for individual paths comes from Markov's inequality:

$$\mathbb{P}(|X| > a) \leq \frac{\mathbb{E}(|X|)}{a} \quad (a > 0)$$

$$\begin{aligned} \text{Quick proof: } \mathbb{E}(|X|) &= \int_{-\infty}^{\infty} |x| \underbrace{\rho(x)}_{\text{probability density}} dx \\ &= \left(\int_{-\infty}^{-a} + \int_{-a}^a + \int_a^{\infty} \right) |x| \rho(x) dx \\ &\geq \left(\int_{-\infty}^{-a} + \int_a^{\infty} \right) |x| \rho(x) dx \\ &\geq a \underbrace{\left(\int_{-\infty}^{-a} + \int_a^{\infty} \right) \rho(x) dx}_{\mathbb{P}(|X| > a)} \end{aligned}$$

So applying Markov to the strong error, we, e.g., get

$$\mathbb{P}(|X(t) - X_n| > (\Delta t)^{\frac{\alpha}{2}}) \leq \frac{\mathbb{E}(|X(t) - X_n|)}{(\Delta t)^{\frac{\alpha}{2}}} \approx c_s \frac{(\Delta t)^\alpha}{(\Delta t)^{\frac{\alpha}{2}}} = c_s (\Delta t)^{\frac{\alpha}{2}}$$

=> Strong error also tells us about error for individual paths

• weak error: $|\mathbb{E}(X(t)) - \mathbb{E}(X_n)| \approx C_w (\Delta t)^\beta$, β = weak order of convergence

note: $|\mathbb{E}(X(t) - X_n)| \leq \mathbb{E}(|X(t) - X_n|)$, so weak error $\leq$ strong error

$$(\int |f(x)| dx) \leq \int |f(x)| dx$$

But weak error does not necessarily tell us something about individual paths.

For example, compare $W(t)$ with $0 \Rightarrow \mathbb{E}(W(t)) - \mathbb{E}(0) = 0 - 0 = 0$,
but $W(t)$ is very different from 0 .

Hints for HW7:

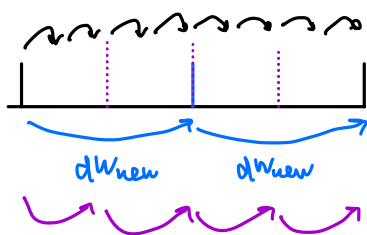
Problem 1: • S_N = Euler-Maruyama after N steps (corresponding to T , i.e., $\Delta t = \frac{T}{N}$)

- By def., Euler-Maruyama is inductive, i.e., it has to be implemented with a "for" loop. In the special case of GBM, one could also use cumprod.

- For strong/weak error: $\#$ is over ensemble, N is varied (# of steps in Euler-Maruyama)

- Note: weak error might be a bit hard to read off; try to fit a line by hand anyway.

- For the error rate, one could start with $N_{\max} = 2^{**}k$ ($k \approx 10$)



$$N = (2, 2^2, 2^3, \dots, \underbrace{2^{**}k}_{N_{\max}})$$

For each realization, create dW of length $N_{\max}$

↳ with that, compute GBM

↳ compute $S_{N_{\max}}$ (with the dW above)

↳ compute $\frac{S_{N_{\max}}}{2}$ by using Euler-Maruyama with coarsened new

$$\frac{dW_{\text{new}}}{2} = (dW_0 + dW_1, dW_1 + dW_2, \dots)$$

↳ repeat till $\frac{N_{\max}}{2^{**}k}$ or sth. smaller