3.4 Itô-lemma

Goal: find a stochastic version of the chain rule

Non-stochastic: given $h(x,t)$, we have $dh = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial t} dt$

Now: $h(W(t),t) = h(x,t)\big|_{x=W(t)}$

Let us write the integral form for the simpler case $h(W(t))$ (no explicit t-dependence):

$$h(W(T)) - h(W(0)) = \sum_{j=0}^{n-1} \Big[\underbrace{h(W(t_{j+1}))} - h(W(t_j)) \Big]$$

Taylor expansion around $W(t_j)$: $h(W_{t_{j+1}}) = h(W(t_j)) + h'(W(t_j))\big(W(t_{j+1}) - W(t_j)\big)$

$$+ \frac{1}{2} h''(W(t_j))\big(W(t_{j+1}) - W(t_j)\big)^2 + \mathcal{O}(\Delta W_j^3)$$

$$\Rightarrow h(W(T)) - h(W(0)) = \sum_{j=0}^{n-1} \Big(h'(W(t_j))\, \Delta W_j + \frac{1}{2} h''(W(t_j))(\Delta W_j)^2 + \mathcal{O}(\Delta W_j^3) \Big)$$

$$\xrightarrow{n\to\infty} \int_0^T h'(W)\, dW + \int_0^T \frac{1}{2} h''(W(t))\, dt$$

remember from last time $(\Delta W_j)^2 \sim \underbrace{\Delta t}_{\text{deterministic}} + \underbrace{\text{Rest}}_{\text{stochastic, but vanishes in the limit: } \sum \Delta t^2 \to 0}\ \mathcal{O}(\Delta t^2)$

$$h(W(T)) - h(W(0)) = \int_0^T \underbrace{\left(\frac{\partial h}{\partial x}\right)(W(t))}_{\text{first derivative, evaluated at } x = W(t)} dW(t) + \int_0^T \frac{1}{2}\left(\frac{\partial^2 h}{\partial x^2}\right)(W(t))\, dt$$

In short-hand notation: $dh = h' dW + \frac{1}{2} h'' dt$

Now if there is also explicit t-dependence, i.e., $h = h(W(t), t)$, then the following Itô formula holds:

$$\boxed{dh = h' dW + \left(\dot{h} + \frac{1}{2} h''\right) dt} \qquad \left(\text{with } \dot{h}(W(t), t) = \frac{\partial h(x,t)}{\partial t}\bigg|_{x=W(t)}\right)$$

$$\left(dh = \left(\frac{\partial h}{\partial x}\right)(W(t))\, dW(t) + \left(\frac{\partial h}{\partial t} + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}\right)(W(t))\, dt\right)$$

Let's consider a few examples:

- $h(W(t), t) = W(t)^2$

  Apply Itô formula: $dh = 2W(t)\, dW(t) + \frac{1}{2} 2\, dt = 2W dW + dt$

  is the SDE (with solution $h = W^2$)

  With that, we can, e.g., compute the expectation value:

$$\mathbb{E}(W(t)^2) - \underbrace{\mathbb{E}(W(0)^2)}_{=0} = 2\int_0^t \underbrace{\mathbb{E}(W(s)\, dW(s))}_{= \mathbb{E}(W(s))\underbrace{\mathbb{E}(dW(s))}_{=0}} + \mathbb{E}\left(\underbrace{\int_0^t ds}_{=t}\right)$$

independence, like we saw last time

$$\Rightarrow \mathbb{E}(W(t)^2) = t$$

- similar example: $h = W^4$

$$\Rightarrow dh = 4W^3 dW + \tfrac{1}{2} 4 \cdot 3 W^2 dt = 4W^3 dW + 6W^2 dt$$

$$\Rightarrow \mathbb{E}(W(T)^4) = \mathbb{E}\left(6 \int_0^T W(t)^2 dt\right) = 6 \int_0^T \underbrace{\mathbb{E}(W(t)^2)}_{= t \text{, as we computed above}} dt$$

$$= 6 \int_0^T t \, dt = 3T^2$$

similarly one could compute $\mathbb{E}(W^{2n})$

- The Itô formula also allows us to solve SDEs. As example, consider:

$$dh = h^3 dt - h^2 dW \quad , h(0) = 1$$

Itô formula: $dh = h' dW + (\dot{h} + \tfrac{1}{2} h'') dt$

We have now reduced an SDE to $\underbrace{\text{two PDEs}}_{\text{those have no stochasticity any more}}$!

$$\Rightarrow \underbrace{h' = -h^2}\ \text{ and }\ \dot{h} + \tfrac{1}{2} h'' = h^3 \ \text{ by comparison}$$

$$\frac{dh}{dx} = -h^2 \overset{\text{seperation of variables}}{\Longrightarrow} \frac{dh}{-h^2} = dx \overset{\text{integrate}}{\Longrightarrow} \underbrace{\int_0^x \frac{dh}{-h^2}} = \underbrace{\int dx}_{= x}$$

$$= h^{-1} \Big|_0^x$$

$$= \frac{1}{h(x)} - \underbrace{\frac{1}{h(0)}}_{=1} = \frac{1}{h(x)} - 1$$

$$\Rightarrow \frac{1}{h(x)} = x + 1 \Rightarrow h(x) = \frac{1}{x+1}$$

$$\Rightarrow \tfrac{1}{2} h'' = \tfrac{1}{2} \left((x+1)^{-1}\right)'' = \tfrac{1}{2} (-1)(-2) (x+1)^{-3} = (x+1)^{-3} = h(x)^3$$, so solution has no explicit time-dependence and is compatible with second eq. above

$$\Rightarrow h(W(t)) = \frac{1}{W(t)+1} \qquad (\text{has singularities for finite } t)$$

For the more general version of the Itô formula, we consider the class of stochastic processes that are solutions to

$$\boxed{dX(t) = f(X(t),t)\,dt + g(X(t),t)\,dW(t)}$$

Solutions $X(t)$ are called **Itô-process.**

For example: $dS = \mu S\,dt + \sigma S\,dW$ (GBM) is an Itô process

So now, we want to look at $F(X(t),t)$ and find an expression for dF.

Similar to before:

$$dF(X,t) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}\underbrace{dX}_{dX = f\,dt + g\,dW} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dX)^2 + \underbrace{\frac{1}{2}\frac{\partial^2 F}{\partial t^2}(dt)^2 + \frac{\partial^2 F}{\partial x \partial t}dX\,dt}_{\to 0 \text{ if integrated}}$$

$$(dX)^2 = (f\,dt + g\,dW)^2 = \underbrace{f^2(dt)^2 + 2fg\,dt\,dW}_{\to 0 \text{ if integrated}} + g^2\underbrace{(dW)^2}_{\to dt}$$

$$\left(\sum_{i=1}^{n}\frac{1}{n^2} \sim \frac{1}{n} \to 0\right) \qquad \left(\sum_{i=1}^{n}\frac{1}{n}\frac{1}{\sqrt{n}} \sim \frac{1}{\sqrt{n}} \to 0\right)$$

$$\Rightarrow \boxed{dF = \left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}f + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}g^2\right]dt + g\frac{\partial F}{\partial x}dW}$$

This is called $\boxed{\text{Itô lemma.}}$

Remark: For $f=0$, $g=1$, we get $X=W$, and in this case

$$dF = \left[\frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \right] dt + \frac{\partial F}{\partial x} dW \quad \text{(just as before)}$$

Let us now discuss the application of Itô's lemma to GBM:

We defined GBM as $S(W(t),t) = e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}$ $\quad (S_0 = 1)$

$\left(S(x,t) = e^{(\mu - \frac{\sigma^2}{2})t + \sigma x} \right)$

With Itô, it satisfies the SDE $dS = \left[\frac{\partial S}{\partial t} + \frac{1}{2} \frac{\partial^2 S}{\partial x^2} \right] dt + \frac{\partial S}{\partial x} dW$

$$= \left[(\mu - \tfrac{\sigma^2}{2}) S + \tfrac{1}{2} \sigma^2 S \right] dt + \sigma S dW$$

$$\Rightarrow \boxed{dS = \underbrace{\mu S}_{f} dt + \underbrace{\sigma S}_{g} dW}$$

With that, we can, e.g., compute $\mathbb{E}(S(t)^n)$.

Take $F(S(t),t) = S(t)^n$ and apply Itô Lemma:

$$\Rightarrow dS^n = \left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} f + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} g^2 \right] dt + g \frac{\partial F}{\partial x} dW$$

$$= \left[0 + n S^{n-1} \mu S + \frac{1}{2} n(n-1) S^{n-2} (\sigma S)^2 \right] dt + \sigma S n S^{n-1} dW$$

$$= \underbrace{\left[n\mu + \frac{1}{2} \sigma^2 n(n-1) \right] S^n}_{f_n(S^n)} dt + \underbrace{\sigma n S^n}_{g_n(S^n)} dW$$

$\Rightarrow$ S^n is again a GBM with different parameters!

$$\Rightarrow \mathbb{E}(S(T)^n) - \underbrace{\mathbb{E}(S(0)^n)}_{=1} = \left(n\mu + \frac{1}{2}\sigma^2 n(n-1)\right) \int_0^T \mathbb{E}(S(t)^n)\,dt + \sigma n \int_0^T \underbrace{\mathbb{E}(S(t)^n dW(t))}_{=0}$$

$\left(\mathbb{E}(S(t)^n) = k(t) \Rightarrow k(T) - k(0) = (n\mu + \frac{1}{2}\sigma^2 n(n-1)) \int_0^T k(t)\,dt \right.$, or $\left. \frac{dk}{dt} = (n\mu + \frac{1}{2}\sigma^2 n(n-1))\, k \right)$

$$\Rightarrow \mathbb{E}(S(t)^n) = e^{[n\mu + \frac{1}{2}\sigma^2 n(n-1)]t}$$

In particular:

- $\mathbb{E}(S(t)) = e^{\mu t}$
- $\mathrm{Var}(S(t)) = \mathbb{E}(S(t)^2) - \mathbb{E}(S(t))^2$

$$= e^{(2\mu + \sigma^2)t} - e^{2\mu t}$$

$$= e^{2\mu t}\left(e^{\sigma^2 t} - 1\right) \quad \left(\approx \sigma^2 t \text{ for small } t\right)$$