Stochastic Modeling and Financial Mathematics

Week 11

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Lecture notes from Fall 2025

One remark:

$$S(W(t),t) = e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}$$

$$S(x,t) = e^{(\mu - \frac{\sigma^2}{2})t + \sigma x}, \quad \text{later } x = W(t)$$

Derivative of S w.r.t. first variable: $\left.\frac{\partial S}{\partial x}\right|_{x=W(t)} = \sigma S(x,t)\Big|_{x=W(t)} = \sigma S(W(t),t)$

(It would be strange to write $\frac{\partial S}{\partial W(t)} = \sigma S(W(t),t)$

Itô's lemma: $dF = \left(\dot{F} + F' f + \frac{1}{2} F'' g^2\right) dt + g F' \, dW(t)$

partial der. w.r.t. second var

partial der. w.r.t. 1st var

$F(x,t)$

$F(X(t),t)$

Quiz 6 Problem 1:

Itô process: $dX = f\,dt + g\,dW$

Itô's lemma: $dF = \left(\dot{F} + F' f + \frac{1}{2} F'' g^2\right) dt + g F' \, dW$

a) $X(t) = W(t)$, i.e., $f = 0$ and $g = 1$ $\quad (dX = dW \Rightarrow X(t) = W(t))$

$$\Rightarrow dF = \left(\dot{F} + \frac{1}{2}F''\right)dt + F'\,dW \quad \text{for } F(W(t),t)$$

b) $F(W(t),t) = \underline{S(W(t),t) = e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}}$ $\quad$ (GBM)

satisfies $\underline{dS} = \left[\left(\mu - \frac{\sigma^2}{2}\right)S + \frac{1}{2}\sigma^2 S\right]dt + \sigma S\,dW$

$$= \underline{\mu S\,dt + \sigma S\,dW} \qquad \text{(SDE for GBM)}$$

c) $X(t) = S(t)$ satisfies $dS = \underbrace{\mu S}_{f}\,dt + \underbrace{\sigma S}_{g}\,dW$

$F(S(t),t) = S(t)^2 \qquad (F(x,t) = x^2)$

Itô: $$dF = \left(\dot{F} + F'f + \frac{1}{2}F''g^2\right)dt + gF'\,dW$$

$$= \left(\dot{F} + F'\mu S + \frac{1}{2}F''\sigma^2 S^2\right)dt + \sigma S F'\,dW$$

$$= \left(0 + 2S\mu S + \frac{1}{2}2\sigma^2 S^2\right)dt + \sigma S\,2S\,dW$$

$$= (2\mu + \sigma^2)S^2\,dt + 2\sigma S^2\,dW$$

$$\Rightarrow \boxed{dF = (2\mu + \sigma^2)F\,dt + 2\sigma F\,dW}$$

We already know: $dS = \hat{\mu} S\,dt + \hat{\sigma} S\,dW \Rightarrow S(W(t),t) = e^{(\hat{\mu} - \frac{\hat{\sigma}^2}{2})t + \hat{\sigma} W(t)}$

$\hat{\mu} = (2\mu + \sigma^2)$, $\hat{\sigma} = 2\sigma$

$\Rightarrow$ explicitly: $F(W(t),t) = e^{\left(2\mu + \sigma^2 - \frac{(2\sigma)^2}{2}\right)t + 2\sigma W(t)} = e^{(2\mu - \sigma^2)t + 2\sigma W(t)}$

Integral form: $F(W(t),t) - F(W(0),0) = (2\mu + \sigma^2)\int_0^t F\,ds + 2\sigma \int_0^t F\,dW(s)$

$$\underbrace{\mathbb{E}(S(t)^2)}_{A(t)} - 1 = (2\mu + \sigma^2)\int_0^t \underbrace{\mathbb{E}(S^2)}_{A(s)}ds + 2\sigma \int_0^t \mathbb{E}(F)\underbrace{\mathbb{E}(dW(s))}_{=0}$$

$$A(t) - 1 = (2\mu + \sigma^2)\int_0^t A(s)\,ds$$

$$\frac{dA(t)}{dt} = (2\mu + \sigma^2)\,A(t) \Rightarrow A(t) = e^{(2\mu + \sigma^2)t}$$

$\mathbb{E}(dW) = 0$

$$S(t) - S(0) = \int_0^t \mu S\,ds + \int_0^t \sigma S\,dW$$

$$\Rightarrow \mathbb{E}(S(t)) - 1 = \mu \int_0^t \mathbb{E}(S)\,ds + 0$$

$$\Rightarrow \mathbb{E}(S(t)) = e^{\mu t}$$

$$\Rightarrow \mathrm{Var}(S(t)) = \mathbb{E}(S(t)^2) - \mathbb{E}(S(t))^2$$

$$= e^{(2\mu + \sigma^2)t} - e^{2\mu t}$$

$$= e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)$$

Again: Without stochasticity, the differential of $X(W,t)$ is

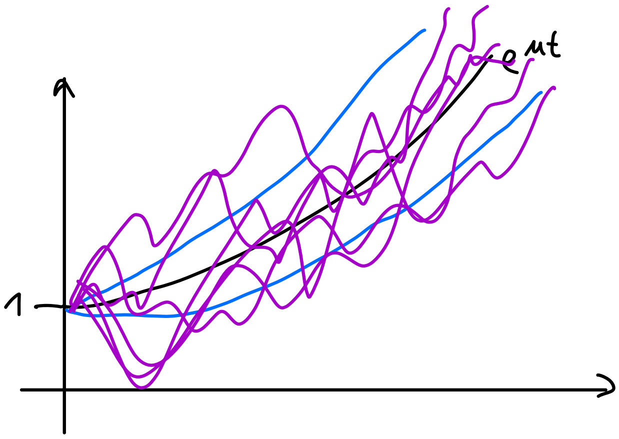
variables

$$dX = \frac{\partial X}{\partial W}dW + \frac{\partial X}{\partial t}dt$$

If $W(t)$ = Brownian motion, there is an extra contribution coming from $(dW)^2 \simeq dt$:

Then $dX = \frac{\partial X}{\partial W}dW + \left(\frac{\partial X}{\partial t} + \frac{1}{2}\frac{\partial^2 X}{\partial W^2}\right)dt$ (simple form of Itô's lemma)

Problem 2:



Problem 3: $dF = \left(\mu + \frac{1}{1+t}\right) F\,dt + \sigma F\,dW$

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for i in range(N):
    F[i+1] = F[i] + (μ + 1/(1+t[i]))*F[i]*dt + σ*F[i]*dW[i]
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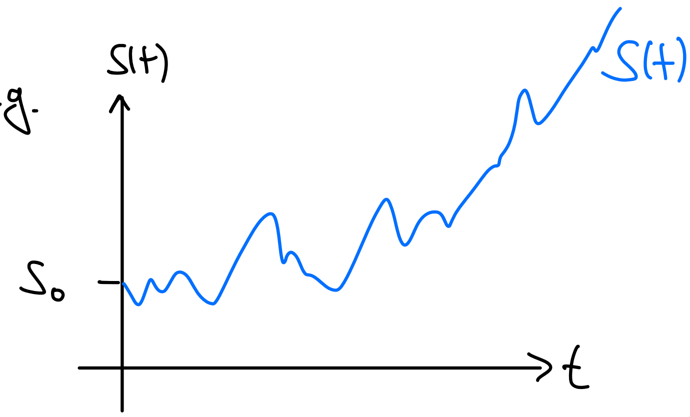
4. Black-Scholes Eq. starting from Cont. Model

We assume the following continuous stock price model:

GBM $\quad S(t) = S_0 \, e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}$

which we know is the solution to $\quad dS = \mu S \, dt + \sigma S \, dW$

E.g.



Previously:
- Start from discrete N-step model
- derive EU call option price (summation formula)
- take $N \to \infty \Rightarrow$ Black-Scholes formula

We know from HW?: Continuum limit of our N-step model is GBM!

So today the question is: How do we get Black-Scholes starting from GBM?

$$
\begin{tikzcd}
N\text{-step model} \arrow[r, "\text{option pricing}"] \arrow[d, shift left, "N \to \infty"] & \text{summation formula} \arrow[d, "N \to \infty"] \\
\text{GBM} \arrow[u, shift left, "\text{discretize}"] \arrow[r, "?", "\text{today}"'] & \text{BS-formula}
\end{tikzcd}
$$

Option price $C(S(t), t)$ (+ depends possibly on parameters r, μ, σ, T, K)

EU calls:

At $t = T$ we know $C(S(T), T) = \max(S(T) - K, 0)$

Idea: Set up replicating portfolio; later $C(S(0), 0)$ = option price at $t = 0$

= portfolio price at $t = 0$

(last time: $C = \overset{\text{bonds}}{x_1} + S \overset{\#\text{of stocks}}{x_2}$)

$$\Pi = \alpha \overset{\text{option price at time } t}{C} + \beta \underset{S(t) = GBM}{S} \quad , \alpha, \beta \text{ tbd.}$$

$\underset{\text{bonds}}{\Pi}$

At the end $\Pi(t) = \Pi(0) e^{rt}$ should be risk-free.

$\Rightarrow \boxed{d\Pi = r \Pi dt}$

$\left(df = rf\, dt \Rightarrow \frac{df}{dt} = rf, \text{ solved by } f(t) = f_0 e^{rt} \right)$

Also Itô: $d\Pi = \alpha dC + \beta dS$ with $C = C(S(t), t)$

$$= \alpha \left(\frac{\partial C}{\partial t} + \underset{\mu S}{f} \frac{\partial C}{\partial x} + \frac{1}{2} \frac{\partial^2 C}{\partial x^2} \underset{g = \sigma S}{g^2} \right) dt + \alpha g \frac{\partial C}{\partial x} dW$$

$$+ \beta \underbrace{dS}_{dS = \mu S dt + \sigma S dW}$$

$$\Rightarrow d\Pi = \alpha\left(\frac{\partial c}{\partial t} + \mu S\frac{\partial c}{\partial x} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 c}{\partial x^2}\right)dt + \beta\mu S\,dt$$

$$+ \underbrace{\left(\beta\sigma S + \alpha\sigma S\frac{\partial c}{\partial x}\right)dW}$$

Must vanish, hence chose $\beta = -\alpha\frac{\partial c}{\partial x}$

$$\Rightarrow d\Pi = \alpha\left(\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 c}{\partial x^2}\right)dt$$

Also: $d\Pi = r\Pi\,dt = r(\alpha C + \beta S)dt = r\left(\alpha C - \alpha\frac{\partial c}{\partial x}S\right)dt$

$$\Rightarrow \boxed{\frac{\partial c}{\partial t} + r\frac{\partial c}{\partial x}S + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial x^2} = rC}$$ Black-Scholes Eq. (PDE)

drift diffusion inhomogeneous

"Initial" cond.: $C(x,T) = \max(x-K,0)$

$\Rightarrow$ Solving this "backwards" yields $C(x,0)$, the option price!