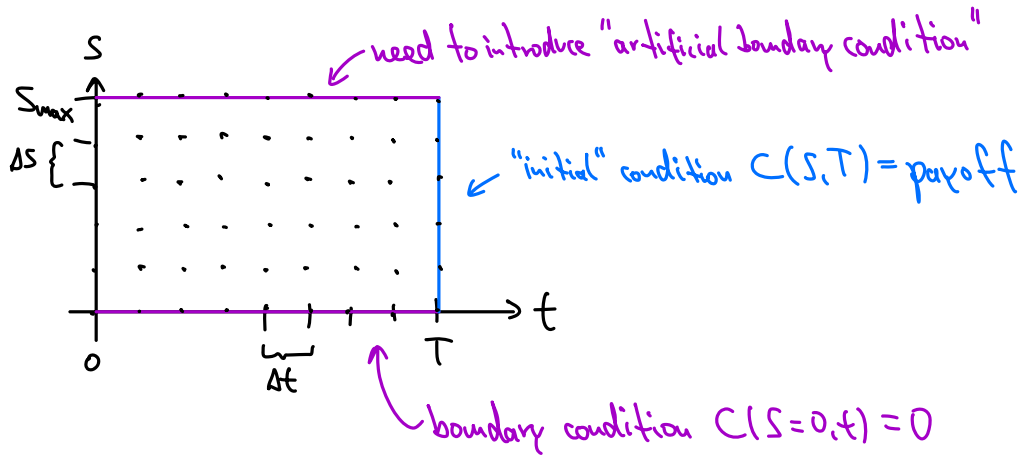


4.3 Discrete Finite Differences

In order to solve PDEs such as the Black-Scholes eq., we need to discretize S, t :



What do we use for $C(S_{\max}, t)$? (for European calls)

↳ We know $C(S_{\max}, T) = \max(S_{\max} - K, 0) = S_{\max} - K$ ($S_{\max}$ chosen large enough)

↳ Also: $C(S_{\max}, 0) = S_{\max} - Ke^{-rT}$, since risk vanishes as $S_{\max} \rightarrow \infty$.

$\Rightarrow$ Could choose interpolation $C(S_{\max}, t) = S_{\max} - Ke^{-r(T-t)}$ as boundary condition.

Other (simpler) possibilities:

– $C(S_{\max}, t) = S_{\max} - K$

– $C(S_{\max}, t) = S_{\max}$, justified if $S_{\max} \gg K$

We have discretized $[0, S_{\max}] \times [0, T]$ into a grid:

– M steps of size $\Delta t = \frac{T}{M}$, $t_j = j \cdot \Delta t$

– N steps of size $\Delta S = \frac{S_{\max}}{N}$, $S_i = i \cdot \Delta S$

We abbreviate $C(s_i, t_j) = C_{i \leftarrow s}^j \leftarrow \text{time}$

Then:

$$\frac{\partial C_i^j}{\partial t} = \frac{C_{i+1}^j - C_i^j}{\Delta t} + \underbrace{O(\Delta t)}$$

terms of order Δt , i.e. $\frac{O(\Delta t)}{\Delta t} \xrightarrow{\Delta t \rightarrow 0} \text{const}$

$$\left(\text{Taylor: } \underbrace{C(s_i, t_j + \Delta t)}_{C_{i+1}^j} = \underbrace{C(s_i, t_j)}_{C_i^j} + \frac{\partial C_i^j}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 C_i^j}{\partial t^2} (\Delta t)^2 + O(\Delta t^3) \right)$$

For the s -derivative, we could choose

$$\frac{\partial C_i^j}{\partial s} = \frac{C_{i+1}^j - C_i^j}{\Delta s} + O(\Delta s) \quad , \text{ but here one can do better}$$

Taylor:

$$C(s_i + \Delta s) = C(s_i) + \frac{\partial C(s_i)}{\partial s} \Delta s + \frac{1}{2} \frac{\partial^2 C(s_i)}{\partial s^2} (\Delta s)^2 + \frac{1}{6} \frac{\partial^3 C_i}{\partial s^3} (\Delta s)^3 + O(\Delta s^4) \quad (1)$$

$$C(s_i - \Delta s) = C(s_i) - \frac{\partial C(s_i)}{\partial s} \Delta s + \frac{1}{2} \frac{\partial^2 C(s_i)}{\partial s^2} (\Delta s)^2 - \frac{1}{6} \frac{\partial^3 C_i}{\partial s^3} (\Delta s)^3 + O(\Delta s^4) \quad (2)$$

$$(1) - (2) \Rightarrow C(s_i + \Delta s) - C(s_i - \Delta s) = 2 \frac{\partial C(s_i)}{\partial s} \Delta s + O(\Delta s^3)$$

$\Rightarrow$ The centered derivative $\frac{\partial C_i^j}{\partial s} = \frac{C_{i+1}^j - C_{i-1}^j}{2 \Delta s} + O(\Delta s^2)$ improves the error

Second derivative: $(1) + (2) \Rightarrow \frac{\partial^2 C_i^j}{\partial s^2} = \frac{C_{i+1}^j - 2C_i^j + C_{i-1}^j}{(\Delta s)^2} + O(\Delta s^2)$

↑
same error as centralized first derivative

6.4 Stability of Time-stepping Methods

Stability: Does the approximating (discretized) solution converge to the true solution?

Let us just consider the simple example of exponential decay:

$$\frac{dy}{dt} = -\lambda y, \lambda > 0 \Rightarrow y(t) = y_0 e^{-\lambda t}$$

There are several ways to discretize the equation; we discuss the two most important ones:

• Explicit Euler Method: $\frac{y^{j+1} - y^j}{\Delta t} = -\lambda y^j$ (rhs evaluated at j)

$$\Rightarrow y^{j+1} = y^j - \lambda \Delta t y^j = (1 - \lambda \Delta t) y^j$$

$$\Rightarrow y^M = (1 - \lambda \Delta t)^M y_0$$

$$\left(\text{check: } \lim_{M \rightarrow \infty} (1 - \lambda \underbrace{\Delta t}_M)^M y_0 = e^{-\lambda t} y_0 \right)$$

In particular: Given y_0 , we want the y^j to be decreasing:

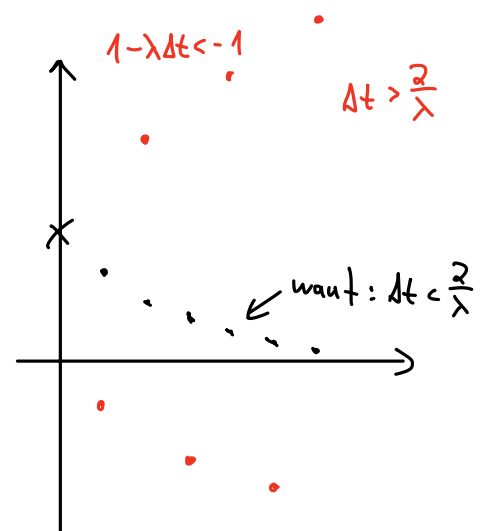
So we need that $|1 - \lambda \Delta t| < 1$ i.e.,

$$\bullet 1 - \lambda \Delta t < 1 \Rightarrow \lambda \Delta t > 0 \text{ holds anyway}$$

$$\bullet -(1 - \lambda \Delta t) < 1 \Rightarrow \lambda \Delta t < 2 \Rightarrow \Delta t < \frac{2}{\lambda}$$

this is a condition on stability

$$\Rightarrow \text{Here, we need to choose } \Delta t \text{ small enough: s.t. } \Delta t < \frac{2}{\lambda}$$



- General conclusion:
- easy to write down and to solve for y^{j+1} (in terms of y^j), even for eq.s such as $\frac{y^{j+1} - y^j}{\Delta t} = f(y^j)$
 - there are usually stability conditions, e.g., Δt must be chosen small enough (or later, Δt must be small enough compared to Δs or Δx)

Implicit Euler Method: $\frac{y^{j+1} - y^j}{\Delta t} = -\lambda y^{j+1}$ (rhs is evaluated at $j+1$)

$$\Rightarrow y^{j+1} - y^j = -\lambda \Delta t y^{j+1} \Rightarrow y^{j+1} = \frac{1}{1 + \lambda \Delta t} y^j$$

$$\Rightarrow y^n = \left(\frac{1}{1 + \lambda \Delta t} \right)^n y_0$$

So here, we need that $\left| \frac{1}{1 + \lambda \Delta t} \right| < 1$, which always holds for $\lambda > 0$.

So here the scheme converges unconditionally!

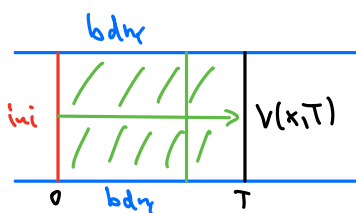
If $\frac{y^{j+1} - y^j}{\Delta t} = f(y^{j+1})$ one needs to (locally) invert f

- General conclusion:
- might not be so easy to solve for y^{j+1} (in terms of y^j) and computationally less efficient than explicit schemes
 - implicit schemes are usually unconditionally stable

4.5 Application to the Heat Equation

Heat eq.: $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$, with initial value $V(x, 0) = V_{\text{ini}}(x)$

say, with boundary conditions $V(x_0, t) = V_0$, $V(x_{\text{max}}, t) = V_{\text{max}}$



(for B.-S. eq., it is the other way around: given $V(x, T)$, solve for $V(x, 0)$)

$$\frac{\partial V}{\partial t}(x_{i,t_j}) = \frac{V(x_{i,t_{j+1}}) - V(x_{i,t_j})}{\Delta t} + O(\Delta t) \quad \text{explicit}$$

$$\frac{\partial V}{\partial t}(x_{i,t_{j+1}}) = \frac{V(x_{i,t_{j+1}}) - V(x_{i,t_j})}{\Delta t} + O(\Delta t) \quad \text{implicit}$$

$$\text{eg.: } \frac{\partial V}{\partial t}(x,t) = \frac{\partial^2 V}{\partial x^2}(x,t)$$

$$\frac{\partial^2 V}{\partial x^2}(x_{i,t}) = \frac{V(x_{i+1,t}) - 2V(x_{i,t}) + V(x_{i-1,t}))}{(\Delta x)^2} + O(\Delta x^2)$$

$$V(x_{i,t_j}) =: V_i^j$$

Explicit scheme: $\frac{V_i^{j+1} - V_i^j}{\Delta t} = \frac{V_{i+1}^j - 2V_i^j + V_{i-1}^j}{(\Delta x)^2}$ (here: given V_i^j , go to V_i^{j+1})

$$\Rightarrow V_i^{j+1} = \frac{\Delta t}{(\Delta x)^2} V_{i+1}^j + \left(1 - 2 \frac{\Delta t}{(\Delta x)^2}\right) V_i^j + \frac{\Delta t}{(\Delta x)^2} V_{i-1}^j$$

we will see numerically that here $\frac{\Delta t}{(\Delta x)^2} < \text{const}$ is a condition for stability

Implicit scheme: $\frac{V_i^{j+1} - V_i^j}{\Delta t} = \frac{V_{i+1}^{j+1} - 2V_i^{j+1} + V_{i-1}^{j+1}}{(\Delta x)^2}$

$$V_i^j = - \underbrace{\frac{\Delta t}{(\Delta x)^2}}_{=: a} V_{i+1}^{j+1} + \left(1 + 2 \frac{\Delta t}{(\Delta x)^2}\right) V_i^{j+1} - \frac{\Delta t}{(\Delta x)^2} V_{i-1}^{j+1}$$

Want to write this as $\vec{V}^j = A \vec{V}^{j+1}$
↙
vector in i

In matrix notation:

← boundary condition still missing

$$\underbrace{\begin{pmatrix} (1+2a) & -a & & 0 \\ -a & (1+2a) & -a & \\ & -a & \ddots & \vdots \\ 0 & & \ddots & -a \\ & & & -a & (1+2a) \end{pmatrix}}_A \begin{pmatrix} V_1^{j+1} \\ V_2^{j+1} \\ \vdots \\ V_n^{j+1} \end{pmatrix} = \begin{pmatrix} V_1^j \\ V_2^j \\ \vdots \\ V_n^j \end{pmatrix}$$

tridiagonal matrix

$\Rightarrow$ write solution as $\vec{V}^{j+1} = A^{-1} \vec{V}^j$, i.e., need to invert A ;
 or equivalently, solve the system of linear equations $\vec{V}^j = A \vec{V}^{j+1}$, e.g., by Gaussian elimination

Boundary condition:

• $V(x_0, t) = V_0$, i.e., $V_0 = V_0^{j+1} \forall j$:

$$V_1^j = -a V_2^{j+1} + (1+2a) V_1^{j+1} - \underbrace{a V_0^{j+1}}_{\text{given boundary condition}}$$

• $V(x_{\max}, t) = V_{\max}$, i.e., $V_{n+1}^{j+1} = V_{\max} \forall j$:

$$V_n^j = -a \underbrace{V_{n+1}^{j+1}}_{\text{given boundary condition}} + (1+2a) V_n^{j+1} - a V_{n-1}^{j+1}$$

So the right equation including the boundary conditions is:

$$\begin{pmatrix} (1+2a) & -a & & & \\ -a & (1+2a) & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & -a & (1+2a) \end{pmatrix} \underbrace{\begin{pmatrix} V_1^{j+1} \\ V_2^{j+1} \\ \vdots \\ V_n^{j+1} \end{pmatrix}}_{\text{what I used to solve for}} = \underbrace{\begin{pmatrix} V_1^j + a V_0^{j+1} \\ V_2^j \\ \vdots \\ V_n^j + a V_{n+1}^{j+1} \end{pmatrix}}_{\text{given}}$$

given bdy conditions

or

$$A \vec{V}^{j+1} = \vec{V}^j + \begin{pmatrix} a V_0^{j+1} \\ 0 \\ \vdots \\ 0 \\ a V_{n+1}^{j+1} \end{pmatrix}$$