Stochastic Modeling and Financial Mathematics

Week 13

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Lecture notes from Fall 2025

Summary of class:

Part 1: Discrete models

- Basics of interest compounding
- Bonds, especially 0-coupon bonds
- Binary models and option pricing
- Basics of binomial trees: European calls: Price $= \mathbb{E}_{\text{risk-free prob.}}(\text{payoff})$

  → direct evaluation

  → Monte-Carlo

- Continuum limit of bin. tree $\xrightarrow{\text{using CLT}}$ Black-Scholes formula

Part 2: Continuum models

- Brownian Motion, Geometric BM

  ↳ how to describe GBM via

  → explicitly: $S(t) = \dots$

  → via SDE: $dS = \dots$

  → via integral eq.: $S(t) - S(0) = \int \dots$

  ↳ numerically: Euler-Maruyama

- Itô's lemma (and how it applies to GBM)
- Option pricing using Itô's lemma

- Option pricing with Black-Scholes PDE

- Numerically solving PDEs: Euler explicit / implicit

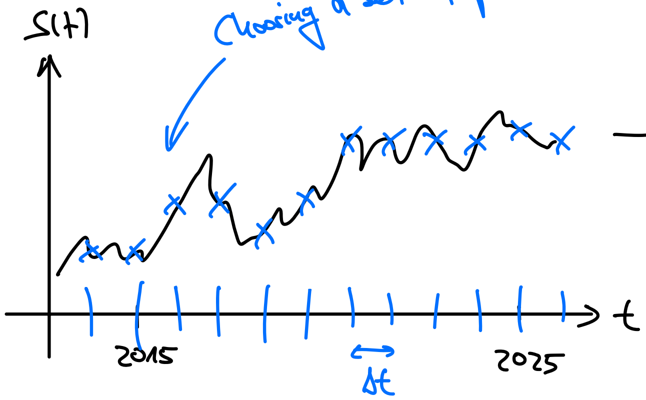
autocorrelation plot

- Estimating the parameters μ, σ from data and checking the normality and independence assumptions

QQ-plot

5. Parameter Estimates for Time Series

Choosing a set of points $(t_i, S(t_i))$ is called <u>sampling</u>



$\longrightarrow$ time series $\times \ldots \times$

Simple way of sampling: Take $t_i - t_{i-1} = \Delta t = \text{const}$

Keep in mind: There is many different ways one can sample.

Our assumption is that $S(t)$ is a GBM: $S(t) = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}$.

Sampling leads to data $(t_i, S(t_i))_{i=1,\ldots,N}$.

Convenient here: Consider log-returns r_i from $S(t_{i+1}) = S(t_i) e^{r_i}$

$$\Rightarrow r_i = \ln \frac{S(t_{i+1})}{S(t_i)} = \ln S(t_{i+1}) - \ln(S(t_i))$$

Now: - What are the r_i's for our model?

- Then: r_i's of our sample?

For GBM: $r_i = \ln S(t_{i+1}) - \ln S(t_i)$
(Model)

$$= \ln S_0 e^{(\mu - \frac{\sigma^2}{2})t_{i+1} + \sigma W(t_{i+1})} - \ln S_0 e^{(\mu - \frac{\sigma^2}{2})t_i + \sigma W(t_i)}$$

$$= (\mu - \tfrac{\sigma^2}{2})t_{i+1} + \sigma W(t_{i+1}) - \left[(\mu - \tfrac{\sigma^2}{2})t_i + \sigma W(t_i)\right]$$

$$\Rightarrow \quad r_i = \left(\mu - \frac{\sigma^2}{2}\right)\underbrace{\left(t_{i+1} - t_i\right)}_{\Delta t_i} + \sigma \underbrace{\left(W(t_{i+1}) - W(t_i)\right)}_{\Delta W_i}$$

Here: assume sampling equally spaced: $\Delta t_i = \Delta t = \text{const}$

$$\Rightarrow \quad r_i = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma \Delta W$$

The r_i's (in our model) are: • Normally distributed

• independent

Need to check later for our sample

With $\mathbb{E}(r_i) = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma \underbrace{\mathbb{E}(\Delta W)}_{=0} = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t$

$$\mathrm{Var}(r_i) = \mathrm{Var}\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma \Delta W\right) = \mathrm{Var}(\sigma \Delta W) = \sigma^2 \mathrm{Var}(\Delta W)$$

$$= \sigma^2 \Delta t$$

$$\Rightarrow \quad r_i \sim \mathcal{N}\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t, \sigma\sqrt{\Delta t}\right)$$

std. deviation

Next: Sample (Data): • expectation/mean/average $\bar{r} = \frac{1}{N}\sum_{i=1}^{N} r_i$

(recall $r_i = \ln S_{i+1} - \ln S_i$)

data

• variance: $\sigma_r^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(r_i - \bar{r}\right)^2$

taking here $\frac{1}{N-1}$ instead of $\frac{1}{N}$ is called "unbiased sample variance". You get this from taking an average over all possible ways of sampling

random variables w.r.t. different sampling possibilities

Now set $\mathbb{E}(r_i) = \bar{r}$, $\mathrm{Var}(r_i) = \sigma_r^2$.

$$\Rightarrow \sigma_r^2 = \hat{\sigma}^2 \Delta t \Rightarrow \boxed{\hat{\sigma} = \sqrt{\frac{\sigma_r^2}{\Delta t}} = \frac{\sigma_r}{\sqrt{\Delta t}} = \text{estimated theoretical sigma}}$$

$$\bar{r} = \left(\hat{\mu} - \frac{\hat{\sigma}^2}{2}\right)\Delta t \Rightarrow \boxed{\hat{\mu} = \frac{\bar{r}}{\Delta t} + \frac{\hat{\sigma}^2}{2} = \frac{\bar{r}}{\Delta t} + \frac{\sigma_r^2}{2\Delta t} = \text{estimated theoretical mu}}$$

Important point (that we will check numerically):

- One can show (see HW) that $\mathrm{Var}(\hat{\sigma}) = \frac{\mathbb{E}(\hat{\sigma})^2}{N}$. This is good: our estimate becomes better the more sample points we take.

- But: $\mathrm{Var}(\hat{\mu})$ does not decrease with N.
  ↳ It does become better the larger T.

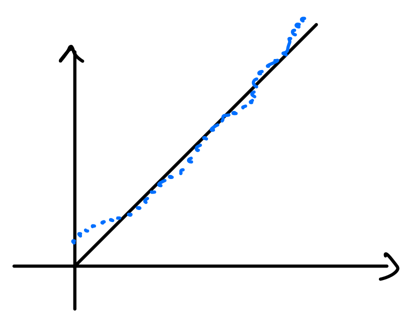
Upshot: σ can be reliably estimated, the μ not.

Luckily, we only need σ for the option pricing!

Still need to check:

- Assumption of Normality: QQ plot

  plot rescaled and sorted r_i's against normal dist.:



- Assumption of Independence:

Autocorrelation Function (ACF): $K_\ell = \dfrac{\mathrm{Cov}(r_i, r_{i-\ell})}{\sqrt{\mathrm{Var}(r_i)}\sqrt{\mathrm{Var}(r_{i-\ell})}}$

(ℓ is called "lag")

$K_0 = 1$ and $|K_\ell| \leq 1$ in general.

- $K_\ell = 1$ perfect correlation
- $K_\ell = -1$ perfect anticorrelation
- $K_\ell \approx 0$ uncorrelated

In practice: