Week 13: ODEs and PDEs

1. MULTI Single

   Find the solution to $y'' + 5y' + 6y = 0$ with initial conditions $y(0) = 2$ and $y'(0) = 3$.

   (a) $y(x) = 5e^{-3x} - 2e^{-2x}$.
   (b) $y(x) = e^{-x} + e^{4x}$.
   (c) $y(x) = 3e^{-4x} + 5e^{-3x}$.
   (d) $y(x) = -7e^{-3x} + 9e^{-2x}$.

2. MULTI Single

   Find the general *complex* solution to $y'' - 2\alpha y' + (\alpha^2 + \beta^2)y = 0$, where $\alpha \in \mathbb{R}$ and $\beta > 0$ are parameters.

   (a) $y(x) = e^{\sqrt{\alpha}x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x})$.
   (b) $y(x) = e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x})$.
   (c) $y(x) = e^{\sqrt{\alpha^2+\beta^2}x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x})$.
   (d) $y(x) = e^{\beta x}(c_1 e^{i\alpha x} + c_2 e^{-i\alpha x})$.

3. MULTI Single

   Find the general *real* solution to $y'' - 2\alpha y' + (\alpha^2 + \beta^2)y = 0$, where $\alpha \in \mathbb{R}$ and $\beta > 0$ are parameters.

   (a) $y(x) = Ae^{\beta x}e^{\alpha x + \varphi}$.
   (b) $y(x) = Ae^{\alpha x}\sin(\beta x + \varphi)$.
   (c) $y(x) = Ae^{-\alpha x}e^{\beta x + \varphi}$.
   (d) $y(x) = Ae^{\beta x}\sin(\alpha x + \varphi)$.

4. MULTI Single

   Find the general solution to $y''' - y'' - y' + y = 0$.

   (a) $y(x) = c_1 e^{2x} + c_2 e^{-x}$.
   (b) $y(x) = c_1 \sin(x) + c_2 \cos(x) + c_3 e^{x}$.
   (c) $y(x) = (c_1 + c_2 x)e^{x} + c_3 e^{-x}$.
   (d) $y(x) = c_1 e^{x} + c_2 e^{-x}$.

5. MULTI Single

   How does the real solution to $y'' + y' + y = 0$ behave for very large x?

   (a) $y(x) \to 1$ as $x \to \infty$.
   (b) $y(x) \to 0$ as $x \to \infty$.
   (c) $y(x) \to \infty$ as $x \to \infty$.
   (d) $y(x) \to -\infty$ as $x \to \infty$.

6. MULTI Single

   How does the real solution to $y'' - 2y' + 10y = 0$ behave for very large x?

   (a) $y(x) \to \infty$ as $x \to \infty$.
   (b) $y(x) \to 1$ as $x \to \infty$.
   (c) $y(x)$ oscillates with larger and larger amplitude as $x \to \infty$.

(d) $y(x) \to 0$ as $x \to \infty$.

7. MULTI Single

Which of the following is the general solution to $y'' - 2y' + y = e^x$?

(a) $y(x) = \frac{x^2}{2}e^x$.
(b) $y(x) = (c_1 + c_2 x)e^x + e^x$.
(c) $y(x) = c_1 e^x + c_2 e^{-x} + \frac{x^2}{2}e^x$.
(d) $y(x) = (c_1 + c_2 x)e^x + \frac{x^2}{2}e^x$.

8. MULTI Single

Which of the following is a solution to the one-dimensional wave equation $c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$?

(a) $u(x,t) = x^2 - c^2t^2$.
(b) $u(x,t) = Ae^x \sin(t + \varphi)$ for constants A and φ.
(c) $u(x,t) = f(x - ct) + g(x + ct)$ for arbitrary twice differentiable functions f and g.
(d) $u(x,t) = \cos(x^2 - c^2t^2)$.

9. MULTI Single

Consider the example of the heat conducting rod of length L from class, i.e., consider the PDE $\kappa\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, for $0 < x < L$ and $t > 0$. Now, we assume that the temperatures at the ends of the rod are fixed, i.e., $u(t, x = 0) = T_1$ and $u(t, x = L) = T_2$ for some given $T_1, T_2 > 0$. What is now the general solution to the equation?

(a) $u(x,t) = f(x - T_1 t) + g(x + T_2 t)$ for arbitrary twice differentiable functions f and g.
(b) $u(t,x) = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2\kappa}{L^2}t} \sin(\frac{n\pi}{L}x)$.
(c) $u(t,x) = \frac{(T_2 - T_1)}{L}x + T_1 + \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2\kappa}{L^2}t} \sin(\frac{n\pi}{L}x)$.
(d) $u(t,x) = \frac{(T_2 - T_1)}{L}x + T_1$.

10. MULTI Single

Which of the following is a solution to the one-dimensional wave equation $c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ for $0 < x < L$, $t > 0$, with boundary conditions $u(t, x = 0) = 0 = u(t, x = L)$, and for initial conditions $u(t = 0, x) = f(x)$, $\frac{\partial u}{\partial t}(t = 0, x) = 0$?

(a) $u(t,x) = \sum_{n=1}^{\infty} c_n \cos(\frac{n\pi c}{L}t) \sin(\frac{n\pi}{L}x)$.

(b) $u(t,x) = \sum_{n=1}^{\infty} c_n \cos(\frac{n^2\pi^2 c^2}{L^2} t) \sin(\frac{n^2\pi^2}{L^2} x)$.

(c) $u(t,x) = \sum_{n=1}^{\infty} c_n e^{\frac{n^2\pi^2 c^2}{L^2} t} \sin(\frac{n^2\pi^2}{L^2} x)$.

(d) $u(t,x) = \sum_{n=1}^{\infty} c_n e^{\frac{n\pi c}{L} t} \sin(\frac{n\pi}{L} x)$.

Total of marks: 10