Elements of Calculus

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Lecture notes from Spring 2025

Session 9
(Week 5 A)

3. Differentiation in One Variable

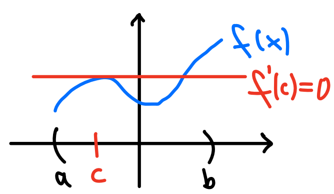
3.2 Theorems and Applications

Topic for Week 5 A: Theorems of Differentiation

Today we discuss several useful theorems that hold for differentiable functions.
(Similar to the Extreme and Intermediate Value Theorems for continuous functions.)

We start with:

Theorem: Let $f:(a,b) \to \mathbb{R}$ be differentiable. If f has a maximum or minimum at $c \in (a,b)$, then $f'(c) = 0$.



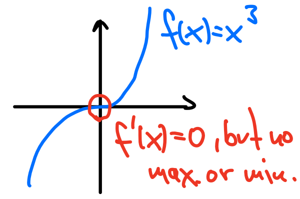
Proof: We consider just the case where f has a maximum at c. Then $f(c+h) \leq f(c)$ for small enough $|h|$. Then:

- If $h > 0$: $\dfrac{f(c+h)-f(c)}{h} \leq 0$, so also $\underbrace{\lim_{h \searrow 0} \dfrac{f(c+h)-f(c)}{h}}_{= f'(c), \text{ since } f \text{ differentiable}} \leq 0$

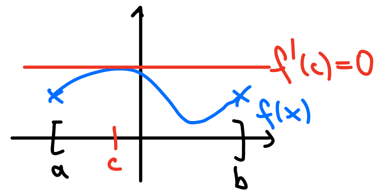
- If $h < 0$: $\dfrac{f(c+h)-f(c)}{h} \geq 0$, so also $\underbrace{\lim_{h \nearrow 0} \dfrac{f(c+h)-f(c)}{h}}_{= f'(c), \text{ since } f \text{ differentiable}} \geq 0$

$\Rightarrow$ Since $f'(c) \leq 0$ and $f'(c) \geq 0$ must hold, we conclude $f'(c) = 0$. $\square$

Note: The other way around does not hold. E.g., $f(x)=x^3$ has $f'(x)=3x^2$, i.e., $f'(0)=0$, but $f(h)>0\ \forall h>0$ and $f(k)<0\ \forall k<0$, so f does not have a max. or min. at 0.



Theorem (Rolle): Let $f:[a,b]\to\mathbb{R}$ be continuous and differentiable on (a,b). If $f(a)=f(b)$, then there exists a $c\in(a,b)$ s.t. $f'(c)=0$.



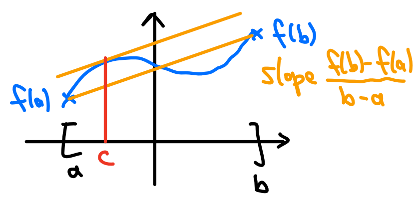
Proof: Since f is continuous on $[a,b]$, the Extreme Value Theorem applies, i.e., f has its max. and min. in $[a,b]$. Since $f(a)=f(b)$, at least one of them lies in (a,b) (unless $f=$ const, where $f'=0$ holds anyway), so the previous theorem applies and proves the result. $\square$

What if $f(a)\neq f(b)$? We shift f linearly and reduce to the previous case.

Def. $h(x):=f(x)-\left(f(a)+(x-a)\dfrac{f(b)-f(a)}{b-a}\right)$. Then $h(a)=0=h(b)$.

So Rolle's thm. implies: $\exists\, c\in(a,b)$ s.t. $0=h'(c)=f'(c)-\dfrac{f(b)-f(a)}{b-a}$

$\Rightarrow f'(c)=\dfrac{f(b)-f(a)}{b-a}$.



We have proven:

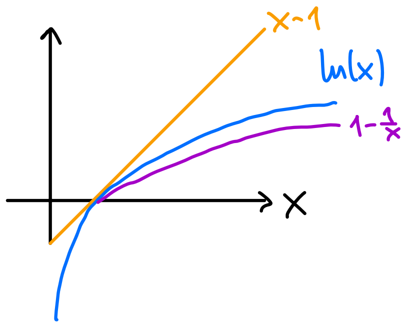
Theorem (Lagrange, or Mean Value Theorem (MVT)): Let $f:[a,b]\to\mathbb{R}$ be continuous and differentiable on (a,b). Then there exists $c\in(a,b)$ s.t. $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Example: Let $f(x)=\ln(x)$ and consider the interval $[1,x]$. Since $f'(x)=\frac{1}{x}$, the MVT says:

$$\exists c\in(1,x) \text{ s.t. } f'(c)=\frac{1}{c}=\frac{\ln(x)-\ln(1)}{x-1}=\frac{\ln(x)}{x-1} \Rightarrow \ln(x)=\frac{x-1}{c}.$$

From this we can infer: • Since $c>1$: $\ln(x)<x-1 \quad \forall x>1$.

• Since $c<x$: $\ln(x)>\frac{x-1}{x}=1-\frac{1}{x} \quad \forall x>1$.



Another implication of the MVT:

- If $f'(x)\geq 0$, then f is increasing on (a,b).
- If $f'(x)\leq 0$, then f is decreasing on (a,b).

Why? $\forall x_1,x_2\in(a,b)$ with $x_1<x_2$ $\exists c\in(x_1,x_2)$ s.t. $\frac{f(x_2)-f(x_1)}{\underbrace{x_2-x_1}_{>0}}=f'(c)$

If ≥ 0, then $f(x_2)\geq f(x_1)$.
If ≤ 0, then $f(x_2)\leq f(x_1)$.

Next: MVT for two functions $\Rightarrow$ Use $g(b)$ and $g(a)$ instead of b and a above.

Def. $\tilde{h}(x):=f(x)-\left(f(a)+(g(x)-g(a))\frac{f(b)-f(a)}{g(b)-g(a)}\right)$. Then $\tilde{h}(a)=0=\tilde{h}(b)$.

So Rolle's thm. implies: $\exists c\in(a,b)$ s.t. $0=\tilde{h}'(c)=f'(c)-g'(c)\frac{f(b)-f(a)}{g(b)-g(a)}$.

$$\Rightarrow \frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)} \quad (\text{assuming } g'(c)\neq 0).$$

Can $g(b)=g(a)$? Not if we assume $g'(x)\neq 0 \quad \forall x\in(a,b)$.

It follows:

Theorem (Cauchy): Let $f, g : [a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b) with $g'(x) \neq 0$ $\forall x \in (a,b)$. Then $g(a) \neq g(b)$ and $\exists c \in (a,b)$ s.t. $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.

An important consequence is:

Theorem (L'Hôpital's Rule): Let $f, g : (a,b) \to \mathbb{R}$ be differentiable with $g'(x) \neq 0$ $\forall x \in (a,b)$.

If $\exists c \in (a,b)$ s.t. $\lim_{x \to c} f(x) = 0 = \lim_{x \to c} g(x)$, then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$.

This rule can be generalized to:

- one-sided limits
- $f(x), g(x) \longrightarrow \infty$ (instead of 0)
- $c = \infty$
- f, g not differentiable in c but $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists

Example: $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$.