Elements of Calculus

Prof. Sören Petrat, Constructor University

Lecture notes from Spring 2025

Example Session Week 7

Example Session for:

Week 7 A: Definite Integrals and the Fundamental Theorem of Calculus

Week 7 B: Applications of Integration

Integral Mean-Value Theorem:

If $f:[a,b]\to\mathbb{R}$ is continuous, then there exists $z\in[a,b]$ s.t. $\frac{1}{b-a}\int_a^b f(x)dx = f(z)$.

Proof:

Let $m := \min_{x\in[a,b]} f(x)$

$M := \max_{x\in[a,b]} f(x)$

the limits exist bc. f is cont. on a closed bounded interval (extreme value thm.)

$\Rightarrow m \le f(x) \le M$ (*)

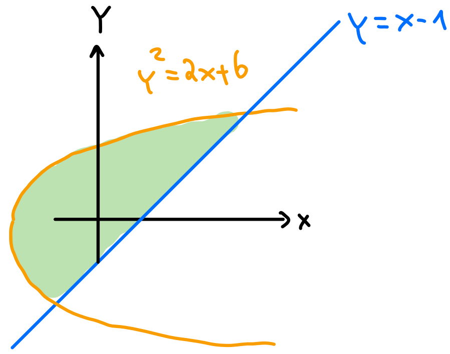
$\Rightarrow \underbrace{\int_a^b m\,dx}_{=(b-a)m} \le \int_a^b f(x)dx \le \underbrace{\int_a^b M\,dx}_{=(b-a)M}$ by Theorem (ii), Session 16.

$\Rightarrow m \le \frac{1}{(b-a)}\int_a^b f(x)dx \le M$

From (*) and the intermediate value thm. we know that f assumes every value between m and M, so in particular there is a $z\in[a,b]$ s.t. $f(z) = \frac{1}{b-a}\int_a^b f(x)dx$. $\square$

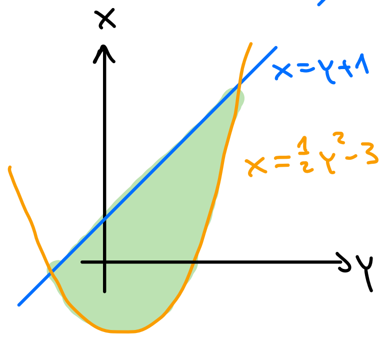
Area between Curves:

How about area between $\underbrace{y^2=2x+6}_{y \text{ is not a fct. of } x \text{ here}}$ and $y=x-1$?



$\Rightarrow$ Express x as fct. of y!

$\Rightarrow \quad x=\frac{1}{2}y^2-3$ and $x=y+1$:



Points of intersection: $\frac{1}{2}y^2-3=y+1 \quad \Rightarrow \quad y^2-2y-8=0$

$$\Rightarrow \quad y = 1 \pm \sqrt{1+8} \quad \Rightarrow \quad y=-2=:a \text{ and } y=4=:b$$

$$\Rightarrow \text{area } A=\int_{-2}^{4}\left(y+1-\left(\tfrac{1}{2}y^2-3\right)\right)dy = \int_{-2}^{4}\left(-\tfrac{1}{2}y^2+y+4\right)dy$$

$$=\left[-\frac{1}{6}y^3+\frac{1}{2}y^2+4y\right]_{-2}^{4} \quad = -\frac{1}{6}\cdot 64+\frac{1}{2}\cdot 16+16-\left(\frac{1}{6}\cdot 8+\frac{1}{2}\cdot 4-8\right)$$

$$= \dots = 18.$$

Rational Functions

A rational function $R(x)$ is the ratio of two polynomials, i.e., $R(x) = \frac{P(x)}{Q(x)}$ for some polynomials P, Q.

E.g., $R(x) = \frac{1}{x(x-1)}$, or $R(x) = \frac{3x^3-9x^2+12x-8}{x^2-3x+2}$.

Question: How do we integrate rational functions?

We proceed in two steps:

1) If $\underbrace{\deg P}_{\text{degree of the polynomial } P} < \deg Q$, continue with step 2).

If $\deg P \geq \deg Q$, write $\frac{P(x)}{Q(x)} = \frac{\tilde{P}(x)}{Q(x)} + \tilde{\tilde{P}}(x)$ with $\tilde{P}$ and $\tilde{\tilde{P}}$ polynomials with $\deg \tilde{P} < \deg Q$.

Ex.: • For $\frac{1}{x(x-1)}$ we already have $\deg P = 0 < \deg Q = 2$.

• For $\frac{3x^3-9x^2+12x-8}{x^2-3x+2}$ we use the following trick:

We write
$$\frac{3x^3-9x^2+12x-8}{x^2-3x+2} = \frac{3x^3-9x^2+12x-8 \overbrace{-3x(x^2-3x+2)+3x(x^2-3x+2)}^{=0}}{x^2-3x+2}$$

$$= \frac{-9x^2+12x-8-3x(-3x+2)}{x^2-3x+2} + 3x$$

$$= \frac{6x-8}{x^2-3x+2} + 3x$$

$\tilde{P}(x), \deg \tilde{P} = 1$

$\deg Q = 2$

= denominator

chosen such that leading order $3x \cdot x^2$ cancels with

2) Factorize $Q(x) = \prod_{k=1}^{n} (x - z_k)^{m_k}$ with z_k the (possibly complex) roots, and m_k the multiplicity of rook z_k.

Then decompose $\frac{\tilde{P}(x)}{Q(x)}$ into partial fractions: $\frac{\tilde{P}(x)}{Q(x)} = \sum_{k=1}^{n} \sum_{j=1}^{m_k} \frac{c_{jk}}{(x-z_k)^j}$.

Then integrate: • $j=1$: $\int \frac{1}{(x-z_k)} dx = \ln(x-z_k)$

• $j>1$: $\int \frac{1}{(x-z_k)^j} dx = \frac{(x-z_k)^{-j+1}}{-j+1}$

Ex.: • $\frac{1}{x(x-1)} = \frac{a}{x} + \frac{b}{x-1} = \frac{a(x-1)+bx}{x(x-1)} = \frac{(a+b)x-a}{x(x-1)}$ and comparing coefficients

yields $a+b=0$, $-a=1$ $\Rightarrow a=-1$, $b=1$.

$$\Rightarrow \int \frac{1}{x(x-1)} dx = \int \frac{-1}{x} dx + \int \frac{1}{x-1} dx = -\ln x + \ln(x-1)$$

• $\frac{6x-8}{x^2-3x+2} = \frac{6x-8}{(x-2)(x-1)} = \frac{a}{x-2} + \frac{b}{x-1} = \frac{a(x-1)+b(x-2)}{(x-2)(x-1)} = \frac{(a+b)x-a-2b}{(x-2)(x-1)}$,

so we need $a+b=6$, $-a-2b=-8$ (a system of linear equations!)

$\Rightarrow b=2$, $a=4$ $\Rightarrow \frac{6x-8}{x^2-3x+2} = \frac{4}{x-2} + \frac{2}{x-1}$

In total, we thus find:

$$\int \frac{3x^3-9x^2+12x-8}{x^2-3x+2} dx = \int \left(\frac{6x-8}{x^2-3x+2} + 3x \right) dx$$

$$= \int \frac{4}{x-2} dx + \int \frac{2}{x-1} dx + \int 3x \, dx$$

$$= 4\ln(x-2) + 2\ln(x-1) + \frac{3}{2}x^2.$$

Note: For complex roots, it might be more convenient to factorize with pairs of complex conjugate roots, i.e., write $(x-z)(x-\bar{z}) = (x-(a+ib))(x-(a-ib))$

$$= (x-a)^2 + b^2$$

, and use this for partial fractions.