Elements of Calculus

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Lecture notes from Spring 2025

Example Session for:

Week 12 A: Lagrange Multipliers

Week 12 B: Least Squares and Gradient Descent

Lagrange Multipliers

Maximize $f(x,y) = x^2 + y^2$ under the constraint $\underbrace{x^2 + xy + y^2 = 6}$.

$$\Rightarrow G(x,y) = x^2 + xy + y^2 - 6 = 0$$

$\Rightarrow$ Lagrangian fct. $L(x,y,\lambda) = f(x,y) - \lambda G(x,y) = x^2 + y^2 - \lambda(x^2 + xy + y^2 - 6)$

$$\Rightarrow \quad \nabla L(x,y,\lambda) = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_\lambda \end{pmatrix} L(x,y,\lambda) = \begin{pmatrix} 2x - \lambda(2x+y) \\ 2y - \lambda(x+2y) \\ -(x^2+xy+y^2-6) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} (\mathrm{I}) \\ (\mathrm{II}) \\ (\mathrm{III}) \end{matrix}$$

$$\left.\begin{matrix} (\mathrm{I}) & \lambda = \dfrac{2x}{2x+y} \\ (\mathrm{II}) & \lambda = \dfrac{2y}{x+2y} \end{matrix}\right\} \Rightarrow \frac{2x}{2x+y} = \frac{2y}{x+2y} \Rightarrow 2x(x+2y) = 2y(2x+y) \Rightarrow 2x^2 + 4xy = 4xy + 2y^2$$

$$\Rightarrow x^2 = y^2 \Rightarrow x = \pm y$$

For $x = y$: $0 = G(x,x) = x^2 + xx + x^2 - 6 \Rightarrow 3x^2 = 6 \Rightarrow x = \pm\sqrt{2}$

For $x=-y$: $0 = 6(x,-x) = x^2 - xx + x^2 - 6 \Rightarrow x^2 = 6 \Rightarrow x = \pm\sqrt{6}$

$\Rightarrow$ The critical points are $(\sqrt{2},\sqrt{2})$, $(-\sqrt{2},-\sqrt{2})$, $(\sqrt{6},-\sqrt{6})$, $(-\sqrt{6},\sqrt{6})$.

We find:

$$\left.\begin{aligned} f(\sqrt{2},\sqrt{2}) &= 2+2=4 \\ f(-\sqrt{2},-\sqrt{2}) &= 2+2=4 \end{aligned}\right\}\ \text{minima}$$

$$\left.\begin{aligned} f(\sqrt{6},-\sqrt{6}) &= 6+6=12 \\ f(-\sqrt{6},\sqrt{6}) &= 6+6=12 \end{aligned}\right\}\ \text{maxima}$$

See also the plot below.

Use https://www.geogebra.org/3d for generating the plots.

The plot shows $f(x, y) = x^2 + y^2$ (red) under the constraint $G(x, y) = x^2 + xy + y^2 - 6 = 0$ (green).