Elements of Calculus

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Lecture notes from Spring 2025

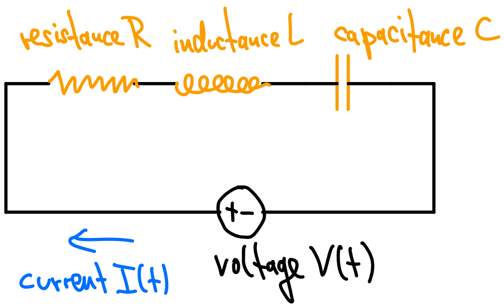
7. ODEs and PDEs

7.1 Linear ODEs

Topic for Week 13 A: Higher Order Linear ODEs

In this session we discuss ODEs that involve higher derivatives but in a linear way.

Example: RLC circuit



The voltage between two points is the same: $V(t) = V_R + V_L + V_C$

$V_R = R\,I(t)$ $\quad V_L = L \frac{dI(t)}{dt}$ $\quad C \frac{dV_C(t)}{dt} = I(t)$

$$\Rightarrow \quad \frac{dV(t)}{dt} = R \frac{dI(t)}{dt} + L \frac{d^2 I(t)}{dt^2} + \frac{1}{C} I(t)$$

Here, $V(t)$ is a given fct., R, L, C are parameters, and we want to solve for $I(t)$

In general, we would like to consider a linear ODE of order n:

$$\boxed{a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x) \frac{dy}{dx} + a_0(x)\, y = f(x)} \quad (*)$$

Here, $a_i(x), i = 1, \dots, n$ and $f(x)$ are given fct.s, and we want to solve for $y(x)$.

Note that the solution will depend on n parameters $c_1, \dots, c_n$ determined by the initial conditions

$y^{n-1}(x_0), y^{n-2}(x_0), \dots, y(x_0)$ (given some initial x_0).

If $f(x)=0$, we call the ODE homogeneous, otherwise inhomogeneous.

First some general considerations.

We start with the homogeneous case $f(x)=0$. Then note that if $y_1(x),\dots,y_n(x)$ solve the ODE $(*)$, then so does $c_1y_1(x)+c_2y_2(x)+\dots+c_ny_n(x)$ (linearity).
This is the general solution if $y_1(x),\dots,y_n(x)$ are linearly independent (because if $y_1,\dots,y_n$ are linearly dependent, then one y_i can be expressed as linear combination of the others, and we only have $n-1$ constants).

$\Rightarrow$ The general sol. to the homogeneous ODE is $y_{hom}(x)=c_1y_1(x)+\dots+c_ny_n(x)$ for $y_1(x),\dots,y_n(x)$ linearly independent sol.s (to the hom. ODE).

For $f(x)\neq 0$, we need to additionally find one particular solution (possibly without any constants).
Let us assume we have such a solution y_{part}. Then we have a general solution $y_{gen}=y_{hom}+y_{part}$.

Suppose $\tilde{y}$ is a different general solution. Then we can always write $\tilde{y}=\underbrace{y_{part}}_{\text{particular sol. from before}}+\underbrace{(\tilde{y}-y_{part})}_{\text{solves the hom. eq.}}$.

We conclude:

Lemma: The general sol. (with n constants) y_{gen} to the inhomogeneous eq. is $y_{part}+y_{hom}$, where y_{part} is any particular sol., and y_{hom} is the general sol. (with n constants) to the homogeneous eq.

Next, let us focus on the case where the coefficients are constant:

$$a_ny^{(n)}+a_{n-1}y^{(n-1)}+\dots+a_1y'+a_0y=f(x)$$

Homogeneous case:

We make the ansatz $y(x) = e^{\lambda x} \Rightarrow y'(x) = \lambda e^{\lambda x}, \dots, y^{(n)}(x) = \lambda^n e^{\lambda x}$

$\Rightarrow$ The ODE becomes $\underbrace{\left(a_n \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1 \lambda + a_0\right)}_{\text{polynomial } \chi(\lambda)} e^{\lambda x} = 0$.

$\Rightarrow y(x)$ is a solution if λ is a zero of the polynomial $\chi(\lambda)$.

Now consider the following cases:

(i) All roots $\lambda_1, \dots, \lambda_n$ are real and distinct. Then the general sol. is $y(x) = \sum_{i=1}^{n} c_i e^{\lambda_i x}$ (since $e^{\lambda_i x}$ for different λ_i are indeed lin. indep.).

(ii) Some roots complex. If there is a complex root $\lambda = a + ib$, then $\bar{\lambda} = a - ib$ (the complex conjugate) is also a root (assuming all coefficient $a_0, \dots, a_n$ are real).

$$\Rightarrow c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x} = e^{ax}\left(c_1 e^{ibx} + c_2 e^{-ibx}\right) = e^{ax}\left((c_1+c_2)\cos bx + i(c_1-c_2)\sin bx\right)$$

Choose the complex parameters c_1, c_2 s.t. d_1, d_2 are real

$$= e^{ax}\left(d_1 \cos bx + d_2 \sin bx\right)$$

By appropriately choosing A, φ or $\hat{A}, \tilde{\varphi}$

$$= A e^{ax} \sin(bx + \varphi)$$

$$= \hat{A} e^{ax} \cos(bx + \tilde{\varphi})$$

(iii) Multiple roots.

Ex.: $y^{(n)} = 0$. Here $\chi(\lambda) = \lambda^n = 0$, i.e., root is zero with multiplicity n. But we easily find the general solution $y(x) = c_1 + c_2 x + c_3 x^2 + \dots + c_n x^{n-1}$.

In general: If a root λ_1 occurs k times $(k > 1)$, it yields a contribution $(c_1 + c_2 x + \dots + c_k x^{k-1}) e^{\lambda_1 x}$ to the general solution.

Inhomogeneous case:

We only need to find one particular solution to $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 = f(x)$.

There is no general method for finding this, but often it can be found using inspection.

First, note that if $f = d_1 f_1 + d_2 f_2$, we can look at f_1 and f_2 seperately:

If y_1 solves $a_n y_1^{(n)} + \dots + a_1 y_1' + a_0 = f_1$ and y_2 solves $a_n y_2^{(n)} + \dots + a_1 y_2' + a_0 = f_2$, then $d_1 y_1 + d_2 y_2$ solves $a_n (d_1 y_1 + d_2 y_2) + \dots + a_1 (d_1 y_1 + d_2 y_2) + a_0 = d_1 f_1 + d_2 f_2 = f$.

We can often use one of the following trial functions:

(i) If $f = ae^{rx}$, try $y_{part}(x) = be^{rx}$.

(ii) If $f = a_1 \sin(rx) + a_2 \cos(rx)$, try $y_{part}(x) = b_1 \sin(rx) + b_2 \cos(rx)$.

(iii) If $f = a_0 + a_1 x + \dots + a_N x^N$, try $y_{part}(x) = b_0 + b_1 x + \dots + b_N x^N$.

Ex.: $y'(x) + 2y(x) = x + 2$

Ansatz: $y_{part}(x) = ax + b \Rightarrow y_{part}'(x) = a \Rightarrow y_{part}' + 2y = \underbrace{a + 2(ax+b) = x + 2}_{\Rightarrow 2a = 1 \text{ and } a + 2b = 2}$

$\Rightarrow a = \frac{1}{2}, b = \frac{3}{4}$ and $y_{part}(x) = \frac{1}{2}x + \frac{3}{4}$.

Sol. to hom. eq. $y' + 2y = 0$: $\chi(\lambda) = \lambda + 2 \Rightarrow \lambda = -2 \Rightarrow y_{hom}(x) = ce^{-2x}$

$\Rightarrow$ The general sol. is $y(x) = ce^{-2x} + \frac{1}{2}x + \frac{3}{4}$.

More examples in the Example Session and the exercises.