Elements of Calculus

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Lecture notes from Spring 2025

Session 26 (Week 13B)

7. ODEs and PDEs

7.2 PDEs

Topic for Week 13B: Partial Differential Equations

An equation that relates a function $u(t, x_1, \ldots, x_n)$ and some of its partial derivatives (possibly of higher order) is called **partial differential equation (PDE)**.

We have already seen one such PDE in Homework 6:

The wave equation $c^2 \nabla^2 u = \frac{\partial^2 u}{\partial t^2}$, where $c > 0$ is a parameter, $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$ and we are looking for a solution $u(t, x_1, \ldots, x_n)$.

For example, for $n=1$ (one-dimension), $u(t,x) = \cos(x - ct)$ is a solution, since

$$c^2 \frac{\partial^2}{\partial x^2} \cos(x-ct) = -c^2 \cos(x-ct) \quad \text{and} \quad \frac{\partial^2}{\partial t^2} \cos(x-ct) = (-c)^2 (-\cos(x-ct)) = -c^2 \cos(x-ct).$$

Basically all laws of physics are formulated as PDEs, e.g., Newtonian mechanics, quantum mechanics (Schrödinger eq.), and electrodynamics (Maxwell's eq.s).

Often, PDEs describe the change in time of a function $u(t, x_1, \ldots, x_n)$ in 1,2,3 dimensional space, but sometimes one looks for equilibrium solutions $f(x_1, \ldots, x_n)$ that do not change in time.

Let us start with giving a list of some of the most important PDEs:

- $\boxed{\text{The wave equation } c^2 \nabla^2 u = \frac{\partial^2 u}{\partial t^2}}$, for $c > 0$.

  Note that $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2} =: \boxed{\Delta \text{ is called Laplacian}}$ (or Laplace operator).

  This eq. describes, e.g., vibrating strings, vibrations in solids/liquids/gas, and electromagnetic waves.

- $\boxed{\text{The heat equation (also "diffusion equation") } \kappa \nabla^2 u = \frac{\partial u}{\partial t}}$, for $\kappa > 0$.

  It describes, e.g., heat flow and diffusion (but is also used for pricing financial instruments, see the Black-Scholes eq.).

- $\boxed{\text{The Laplace equation } \nabla^2 u = 0}$, for a fct. $u(x_1, \dots, x_n)$.

  It describes, e.g., steady-state solutions ($\frac{\partial u}{\partial t} = 0$) of the heat eq., and gravity or electric potentials in free space.

- $\boxed{\text{The Poisson equation } \nabla^2 u = \rho}$, where $\rho(x_1, \dots, x_n)$ is a given function, and we are looking for a solution $u(x_1, \dots, x_n)$.

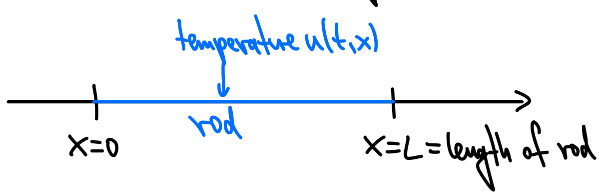
  It describes similar situations as the Laplace eq., but with extra source term ρ present (e.g., a matter or charge density).

- $\boxed{\text{The Schrödinger equation } i\hbar \frac{\partial u}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 u + V(x) u}$, where $\hbar$ = Planck's constant (divided by 2π), m = mass, $V(x_1, \dots, x_n)$ is a given potential fct., and we are looking for a complex-valued fct. $u(t, x_1, \dots, x_n)$. This eq. describes one non-relativistic particle without spin. Many generalizations of the Schrödinger eq. (e.g., to many particles) are available.

Note:

- All equations above except the Poisson eq. are linear.
- The equations above describe three distinct types of physical phenomena: diffusive processes (heat eq.), oscillatory processes (wave and Schrödinger eq.), and steady-state processes (Laplace and Poisson eq.).
- The heat and Schrödinger equations describe an initial value problem in the time variable t. An initial condition $u_0(x_1, \dots, x_n) = u(t=0, x_1, \dots, x_n)$ is given, and the eq.s describe the propagation of u_0 in time. Since the wave eq. is second order in the time derivative, we need to specify initial values $u_0(x_1, \dots, x_n)$ and $\tilde{u}_0(x_1, \dots, x_n) = u'(t=0, x_1, \dots, x_n)$.
- Often the equations above are considered in a finite region of space $\Omega \subset \mathbb{R}^n$. Then one needs to specify boundary conditions on the boundary $\partial\Omega$ of Ω. E.g. if we consider a metal rod $(d=1)$ and the heat eq., we need to specify the temperatures at each end of the rod.

As example, let us investigate a heat conducting solid rod in more detail:



PDE: $\kappa \dfrac{\partial^2 u(t,x)}{\partial x^2} = \dfrac{\partial u(t,x)}{\partial t}$, $0 \leq x \leq L$, $t > 0$.

(The constant $\kappa > 0$ depends on the material of the rod.)

We prescribe the boundary conditions $u(t, x=0) = 0 = u(t, x=L)$ for simplicity.

The eq. is linear and homogeneous, so the idea is to seek many solutions and then take linear combinations of them. We use separation of variables to find some simple basic solutions, i.e., we assume $u(t,x) = f(x)\, g(t)$.

$\Rightarrow \kappa \dfrac{\partial^2 u}{\partial x^2} = \kappa f''(x) g(t)$ and $\dfrac{\partial u}{\partial t} = f(x) g'(t)$

$\Rightarrow$ The PDE becomes $\kappa f''(x) g(t) = f(x) g'(t) \Rightarrow \dfrac{f''(x)}{f(x)} = \dfrac{1}{\kappa} \dfrac{g'(t)}{g(t)}$ (variables separated)

This eq. must be valid for all $0<x<L$ and $t>0$, hence both sides must be equal to the same constant, say $-\lambda$.

$$\Rightarrow \frac{f''(x)}{f(x)} = \frac{1}{\varkappa}\frac{g'(t)}{g(t)} = -\lambda \quad \Rightarrow \text{Two ODEs: } \begin{array}{l} f''(x)+\lambda f(x)=0 \\ g'(t)+\varkappa\lambda g(t)=0 \end{array}$$

We need to satisfy the boundary conditions $u(t,x=0)=f(0)g(t)=0=f(L)g(t)=u(t,x=L)$

$\Rightarrow$ Need $f(0)=0$ and $f(L)=0$.

First: Solve $\underbrace{f''(x)+\lambda f(x)=0}_{\text{homogeneous linear ODE}}$ (which we know from Session 13A), subject to $f(0)=0=f(L)$.

Ansatz: $f(x)=e^{\mu x} \Rightarrow \chi(\mu)=\mu^2+\lambda=0 \Rightarrow \mu=\pm i\sqrt{\lambda}$

$$\Rightarrow f(x)=d_1\cos(\sqrt{\lambda}x)+d_2\sin(\sqrt{\lambda}x)$$

Boundary conditions: $f(0)=d_1=0$

$$f(L)=\underbrace{d_1}_{=0}\cos(\sqrt{\lambda}L)+d_2\sin(\sqrt{\lambda}L)=d_2\sin(\sqrt{\lambda}L)=0$$

Since $d_2=0$ would yield the trivial sol. $f(x)=0$, we need $\sin(\sqrt{\lambda}L)=0$, i.e., $\sqrt{\lambda}=\frac{n\pi x}{L}$, for $n=1,2,3,\dots$.

$\Rightarrow$ We found many solutions $f_n(x)=\sin(\sqrt{\lambda_n}x)$, with $\lambda_n=\frac{n^2\pi^2}{L^2}$, $n\in\mathbb{N}$.

$\Rightarrow$ The eq. for g becomes $g'(t)+\varkappa\frac{n^2\pi^2}{L^2}g(t)=0$, which is solved by $g(t)=e^{-\frac{n^2\pi^2\varkappa}{L^2}t}$.

$\Rightarrow$ We found solutions $u_n(t,x)=e^{-\frac{n^2\pi^2\varkappa}{L}t}\sin(\frac{n\pi}{L}x)$, for any $n\in\mathbb{N}$.

$\Rightarrow u(t,x)=\sum_{n=1}^{\infty}c_n u_n(t,x)=\sum_{n=1}^{\infty}c_n e^{-\frac{n^2\pi^2\varkappa}{L}t}\sin(\frac{n\pi}{L}x)$ with coefficients c_n satisfies the PDE and the boundary conditions $u(t,0)=0=u(t,L)$.

To satisfy the initial condition $u(t=0,x) = u_0(x)$, we need $\sum_{n=1}^{\infty} c_n \sin(\frac{n\pi}{L} x) = u_0(x)$.

This should determine our choice of coefficients c_n.

The expression $\sum_{n=1}^{\infty} c_n \sin(\frac{n\pi}{L} x)$ is called Fourier series. Since it is one of the most important method to solve PDEs and has many more applications, we will study it in more detail next week.