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Lecture notes from Spring 2025

Example Session for:

Week 13 A: Higher Order Linear ODEs

Week 13 B: Partial Differential Equations

Solving a Higher Order Linear ODE

Find the general real solution to $y''(x) - y'(x) + y(x) = \sin(2x)$.

For the particular sol. we make the ansatz $y_{part}(x) = a\sin(2x) + b\cos(2x)$

$$\Rightarrow y_{part}'' - y_{part}' + y = \left(a\sin(2x) + b\cos(2x)\right)'' - \left(a\sin(2x) + b\cos(2x)\right)' + \left(a\sin(2x) + b\cos(2x)\right)$$

$$= \left(-4a\sin(2x) - 4b\cos(2x)\right) - \left(2a\cos(2x) - 2b\sin(2x)\right) + \left(a\sin(2x) + b\cos(2x)\right)$$

$$= (-4a + 2b + a)\sin(2x) + (-4b - 2a + b)\cos(2x)$$

$$= \sin(2x)$$

$\Rightarrow$ Need $-4a + 2b + a = 1 \Rightarrow -3a + 2b = 1$ $\quad \overset{\Uparrow}{} (-3 - \frac{4}{3})a = 1 \Rightarrow a = -\frac{3}{13}$

and $-4b - 2a + b = 0 \Rightarrow -3b - 2a = 0 \Rightarrow b = -\frac{2}{3}a$ $\quad \Downarrow\; b = \frac{2}{13}$

$$\Rightarrow y_{part}(x) = -\frac{3}{13}\sin(2x) + \frac{2}{13}\cos(2x)$$

Hom. sol.: Need to solve $0 = \chi(\lambda) = \lambda^2 - \lambda + 1 = 0$

$$\Rightarrow \lambda_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} - 1} = \frac{1}{2} \pm i\sqrt{\frac{3}{4}}$$

$\Rightarrow y_{hom}(x) = e^{\frac{1}{2}x}\left(d_1 \cos\left(\sqrt{\frac{3}{4}}x\right) + d_2 \sin\left(\sqrt{\frac{3}{4}}x\right)\right)$ for real constants d_1, d_2

or $y_{hom}(x) = A e^{\frac{1}{2}x} \cos\left(\sqrt{\frac{3}{4}}x + \varphi\right)$ for real constants A, φ

$\Rightarrow$ The general sol. is $y(x) = A e^{\frac{1}{2}x} \cos\left(\sqrt{\frac{3}{4}}x + \varphi\right) - \frac{3}{13}\sin(2x) + \frac{2}{13}\cos(2x)$

(for constants A, φ determined by the initial conditions).