Elements of Calculus

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Lecture notes from Spring 2025

Session 28 (Week 14B)

8. Fourier Series

Topic for Week 14B: Convergence and Applications of Fourier Series

Last time: Suppose $f:[0,2\pi]\to\mathbb{C}$ with $f(0)=f(2\pi)$ is Riemann-integrable.

Then we define its Fourier coefficients as $\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx} f(x)\,dx$ and define its Fourier series as $F_f(x) := \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx}$.

Question: In what sense does $F_f(x)$ converge to $f(x)$?

We answer this with help from Linear Algebra: $(e^{ikx})_{k\in\mathbb{Z}}$ should be an $\overbrace{\text{ONB}}^{\text{orthonormal basis}}$ on the vector space of functions $f:[0,2\pi]\to\mathbb{C}$ with $f(0)=f(2\pi)$, with dot/scalar product $\langle f,g\rangle = \frac{1}{2\pi}\int_0^{2\pi}\overline{f(x)}\,g(x)\,dx$ and norm/length $\|f\| = \sqrt{\langle f,f\rangle} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^2dx}$.

$\Rightarrow$ We can hope for $\left\|\sum_{k=-N}^{N}\hat{f}_k e^{ikx} - f(x)\right\| \xrightarrow{N\to\infty} 0$, since $\sum_{k=-\infty}^{\infty}\hat{f}_k e^{ikx}$ is simply a decomposition of f w.r.t. the basis $(e^{ikx})_{k\in\mathbb{Z}}$. The difficulty is of course that $k\in\mathbb{Z}$, i.e., we are dealing with an infinite dimensional vector space.

Let us see how this works. First, note that

$$\|f - \sum_{k=-n}^{n} \hat{f}_k e_k\|^2 = \langle f - \sum_{k=-n}^{n} \hat{f}_k e_k , f - \sum_{k=-n}^{n} \hat{f}_k e_k \rangle$$

$$= \|f\|^2 - \sum_{k=-n}^{n} \Big(\underbrace{\langle f, \hat{f}_k e_k \rangle}_{\substack{= \bar{\hat{f}}_k \langle f, e_k \rangle \\ = \bar{\hat{f}}_k \hat{f}(k) = |\hat{f}_k|^2}} + \underbrace{\langle \hat{f}_k e_k, f \rangle}_{= \hat{f}_k \bar{\hat{f}}_k = |\hat{f}_k|^2} \Big) + \sum_{k=-n}^{n} \sum_{j=-n}^{n} \underbrace{\langle \hat{f}_j e_j, \hat{f}_k e_k \rangle}_{= \hat{f}_j \bar{\hat{f}}_k \delta_{jk}}$$

$$\Rightarrow \|f - \sum_{k=-n}^{n} \hat{f}_k e_k\|^2 = \|f\|^2 - \sum_{k=-n}^{n} |\hat{f}_k|^2$$

As a corollary, we get $\boxed{\text{Bessel's inequality } \sum_{k=-\infty}^{\infty} |\hat{f}_k|^2 \le \|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \, dx .}$

Furthermore: $\underbrace{\|f - \sum_{k=-n}^{n} \hat{f}_k e_k\| \xrightarrow{n\to\infty} 0}_{\text{called "mean-square convergence"}} \iff \|f\|^2 = \sum_{k=-\infty}^{\infty} |\hat{f}_k|^2$ $\boxed{\text{(Parseval identity)}}$

For Example A from last session we find $\|f\|^2 = \frac{1}{2\pi} \int_0^a dx = \frac{a}{2\pi}$ and

$$\sum_{k=-\infty}^{\infty} |\hat{f}_k|^2 = \underbrace{\left(\frac{a}{2\pi}\right)^2}_{=|\hat{f}_0|^2} + \sum_{k\neq 0} \Big| \underbrace{\frac{i}{2\pi k} \left(e^{-ika} - 1\right)}_{\hat{f}_k} \Big|^2$$

$= \dots$ (see HW; use results from Ex. B)

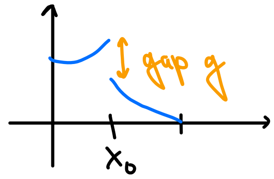
$= \frac{a}{2\pi}$, i.e., the Fourier series converges to f in mean-square.

In general, we can approximate any Riemann-integrable f by such square pulses, which leads to the following result:

Theorem: Let $f: [0, 2\pi] \to \mathbb{C}$, $f(0) = f(2\pi)$ be Riemann-integrable.
Then $\|f - \sum_{k=-n}^{n} \hat{f}_k e^{ikx}\| \xrightarrow{n\to\infty} 0$, i.e., $\sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx} \longrightarrow f(x)$ in mean-square.

(We omit the proof here.)

Let us mention two more properties of the Fourier series. Suppose f is piece-wise continuous and piece-wise differentiable but has a discontinuity at x_0:



Then:

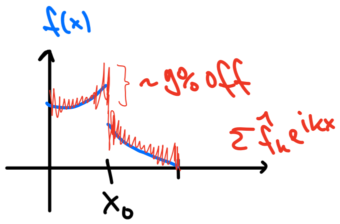
- $$\sum_{k=-n}^{n} \hat{f}_k e^{ikx_0} \xrightarrow{n\to\infty} \frac{1}{2}\Big(\underbrace{\lim_{x\downarrow x_0} f(x)}_{=:f(x_0^+)} + \underbrace{\lim_{x\uparrow x_0} f(x)}_{=:f(x_0^-)}\Big)$$ (as we saw in Example A)

- Let $g := f(x_0^+) - f(x_0^-)$ be the gap at the discontinuity.

Then $$\sum_{k=-n}^{n} \hat{f}_k e^{ikx}\Big|_{x=x_0+\frac{\pi}{n}} \xrightarrow{n\to\infty} f(x_0^+) + g\,c \quad \text{, with } c \approx 0.089\ldots$$

and $$\sum_{k=-n}^{n} \hat{f}_k e^{ikx}\Big|_{x=x_0-\frac{\pi}{n}} \xrightarrow{n\to\infty} f(x_0^-) - g\,c\,.$$

$\Rightarrow$ Near a discontinuity, the Fourier series is ~ 9% off. This is called "Gibbs phenomenon".



We can check this directly for Example A with $a=\pi$: $(f(x_0^+)=0\,,\ g=-1)$

$$\sum_{k=-n}^{n} \hat{f}_k e^{ikx}\Big|_{x=x_0+\frac{\pi}{n}} = \frac{1}{2} + \sum_{\substack{k=1\\ k\text{ odd}}}^{n} \frac{2}{\pi k} \sin\!\left(kx_0 + k\frac{\pi}{n}\right)$$

$x_0=\pi$

$$= \frac{1}{2} + \sum_{\substack{k=1\\ k\text{ odd}}}^{n} \frac{2}{\pi k} \underbrace{\sin\!\left(\pi k + \pi\frac{k}{n}\right)}_{=-\sin\left(\pi\frac{k}{n}\right)}$$

$$= \frac{1}{2} - \underbrace{\sum_{\substack{k=1\\ k\text{ odd}}}^{n} \frac{1}{\left(\frac{n}{2}\right)} \frac{\sin\left(\pi\frac{k}{n}\right)}{\pi\left(\frac{k}{n}\right)}}_{\xrightarrow{n\to\infty} \int_0^1 dx\, \frac{\sin(\pi x)}{\pi x} = \frac{1}{\pi}\int_0^\pi dy\, \frac{\sin y}{y}} \xrightarrow{n\to\infty} \frac{1}{2} - \frac{1}{\pi}\int_0^\pi dy\, \frac{\sin y}{y} \approx -0.089\ldots$$

$\pi x = y$ substitution

Finally, let us come back to the end of the Week 13B session. There, we considered the PDE $\kappa \frac{\partial^2 u(t,x)}{\partial x^2} = \frac{\partial u(t,x)}{\partial t}$, $0 \le x \le L$, $t > 0$, with boundary conditions $u(t, x=0) = 0$ and $u(t, x=L) = 0$, and initial condition $u(t=0, x) = u_0(x)$.

We used separation of variables to conclude that

$$u(t,x) = \sum_{n=1}^{\infty} c_n u_n(t,x) = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2\kappa}{L^2}t} \sin\left(\frac{n\pi}{L}x\right)$$

with some coefficients c_n satisfies the PDE and has the right boundary conditions.

Now: For the initial condition we need $u(t=0,x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) = u_0(x)$ for any given $u_0(x)$ (with $u_0(0) = 0 = u_0(L)$).

From our results above, we now know the c_n must be the Fourier coefficients of $u_0(x)$.

In detail:

$$\int_0^L u_0(x) \sin\left(\frac{k\pi}{L}x\right) dx = \int_0^L \sum_{n=1}^{\infty} c_n \sin\left(\frac{k\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx$$

$$= \sum_{n=1}^{\infty} c_n \underbrace{\int_0^L \sin\left(\frac{k\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx}$$

$$\frac{x}{L} = 2y \quad = 2L \int_0^{1/2} \sin(2\pi k y) \sin(2\pi n y)\, dy = 2L \begin{cases} \frac{1}{4} & \text{if } k=n \quad (k,n \ge 1) \\ 0 & \text{, else} \end{cases}$$

$$= c_k \frac{L}{2}$$

$$\Rightarrow c_k = \frac{2}{L} \int_0^L u_0(x) \sin\left(\frac{k\pi}{L}x\right) dx$$

$\Rightarrow$ The solution to the PDE is

$$u(t,x) = \sum_{n=1}^{\infty} \left[\frac{2}{L} \int_0^L u_0(y) \sin\left(\frac{n\pi}{L}y\right) dy \right] e^{-\frac{n^2\pi^2\kappa}{L^2}t} \sin\left(\frac{n\pi}{L}x\right).$$

A last remark: One can also define the Fourier transform for functions $f:\mathbb{R}\to\mathbb{C}$, or even $f:\mathbb{R}^n\to\mathbb{C}$. Then the Fourier transform is $\hat{f}(k)=(2\pi)^{-\frac{1}{2}}\int_{\mathbb{R}} f(x)e^{ikx}\,dx$ and the function f can be represented as $f(x)=(2\pi)^{-\frac{1}{2}}\int_{\mathbb{R}}\hat{f}(k)e^{-ikx}\,dk$. This is very useful to study PDEs in infinite domains such as $\mathbb{R}^n$, but one needs more advanced techniques for discussing this.