Constructor University
Spring 2025

May 24, 2025

Elements of Calculus
Final Exam

Instructions:

- The exam has 15 multiple choice questions (several answers can be correct!) and 3 longer questions. The total number of points is 113.
- For the multiple choice questions, it is sufficient to mark the final answer(s) only. (No solution steps necessary.) There are no negative points, but of course there are fewer points if wrong answers are selected, or if right answers are not selected.
- For the longer exercises 16, 17, and 18, you need to show your work, i.e., carefully write down the steps of your solution. You will receive points not just based on your final answer, but on the correct steps in your solution.
- No tools or other resources are allowed for this exam. In particular, no notes and no calculators.
- You are free to refer to any results proven in class or the homework sheets unless stated otherwise (and unless the problem is to reproduce a result from class or the homework sheets).

Code of Academic Integrity

I pledge that the answers of this exam are my own work without the assistance of others or the usage of unauthorized material or information.

Sign to confirm that you adhere to the Academic Integrity Code:

Name: ____

Signature: ____

Matric./Student No.: ______________________________

1. **(4 points)** Find the interpolating polynomial $P(x)$ going through the points $(1, 1)$, $(2, 2)$, and $(3, 7)$.
   A. $P(x) = x^2 - 5x + 4$.
   B. $P(x) = 2x^2 - 5x + 4$.
   C. The interpolating polynomial does not exist.
   D. $P(x) = x^2 - 4x + 4$.
   E. $P(x) = 3x^3 + 4x^2 - 5x + 3$.
   F. $P(x) = 6x - 5$.

2. **(6 points)** We consider the geometric series $\sum_{k=0}^{N} x^k$? Which of the following statements are true?
   A. $\sum_{k=0}^{N} x^k = \frac{1-x^{N+1}}{1-x}$.
   B. $\sum_{k=0}^{N} x^k = \frac{1-2x^N}{1-x}$.
   C. $\sum_{k=0}^{N} x^k = \frac{2-x^{N+1}}{1-x}$.
   D. The series converges for $x = 1$.
   E. The series converges for $x = 2$.
   F. The series converges for $x = \frac{1}{2}$.

3. **(4 points)** We consider $f(x) = \frac{2x^2-10x+12}{x-3}$. What is $\lim_{x\to 3} f(x)$?
   A. 1.
   B. ∞.
   C. 2.
   D. 3.
   E. $-\infty$.
   F. 0.

4. **(6 points)** Consider the function

$$f(x) = \sin\left(\frac{1}{x}\right) \quad \text{for } x \neq 0,$$

and $f(0) := 0$. Which of the following is true?

A. $|f(x)| \leq |x|$.
B. $f\left(\frac{1}{\pi}\right) < 0$.
C. f is continuous at 0.
D. $f\left(\frac{1}{\pi}\right) > 0$.
E. f is not continuous at 0.
F. $f\left(\frac{1}{\pi}\right) = 0$.

5. **(4 points)** Calculate the derivative $f'(x)$ of the function

$$f(x) = x^2 \ln(x).$$

A. $f'(x) = x^2 e^x$.
B. $f'(x) = x(1 + 2\ln(x))$.
C. $f'(x) = \frac{2x\ln(x) - x}{(\ln(x))^2}$.
D. $f'(x) = 2x$.
E. $f'(x) = x\ln(x) + x$.
F. $f'(x) = x\ln(x) + 2x$.

6. **(4 points)** Calculate the derivative $f'(x)$ of the function

$$f(x) = (x^3 + 4)^5.$$

A. $f'(x) = 4(x^3 + 4)^4$.
B. $f'(x) = 15x^4 + 240x^{11} + 1420x^8 + 3840x^5 + 3840x^2$.
C. $f'(x) = 15x^2(x^3 + 4)^4$.
D. $f'(x) = \frac{3x^2}{6}(x^3 + 4)^6$.
E. $f'(x) = 2x^2(x^3 + 4)^5$.
F. $f'(x) = \frac{1}{6}(x^3 + 4)^6$.

7. **(4 points)** Which of the following is the Taylor series of $f(x) = \cosh(x) := \frac{e^x+e^{-x}}{2}$ around $x_0 = 0$?

   A. $\sum_{k=0}^{\infty} x^k$.
   B. $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$.
   C. $\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$.
   D. $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}$.
   E. $\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$.
   F. $\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

8. **(4 points)** Compute the integral $\int_0^1 x(x^2+1)^3 \, \mathrm{d}x$. (Hint: One can use integration by substitution.)

   A. $\frac{15}{8}$.
   B. $\frac{2}{3}$.
   C. $\frac{17}{8}$.
   D. $\frac{1}{3}$.
   E. $\frac{17}{3}$.
   F. $\frac{15}{6}$.

9. **(4 points)** Compute the integral $\int_0^1 (x+2)e^x \, \mathrm{d}x$. (Hint: One can use integration by parts.)

   A. $2e+1$.
   B. $\frac{e}{2}$.
   C. $\frac{e^2}{2}$.
   D. e.
   E. $2e-1$.
   F. $2e$.

10. **(4 points)** Compute the integral $\int_0^1 \frac{1}{x^2}\,\mathrm{d}x$. (Hint: This is an improper integral.)
   - A. The integral does not exist.
   - B. 2.
   - C. 1.
   - D. 0.
   - E. -1.
   - F. -2.

11. **(4 points)** Solve the ODE $\frac{\mathrm{d}y}{\mathrm{d}t} = y^2$ with initial condition $y(1) = 1$.
   - A. $y(t) = t^2$.
   - B. $y(t) = \frac{1}{2-t}$.
   - C. $y(t) = \frac{1}{2}e^{-t^2} + \frac{1}{2}$.
   - D. $y(t) = e^{-t}$.
   - E. $y(t) = \frac{1}{3}t^3 + \frac{2}{3}$.
   - F. $y(t) = \frac{1}{(2-t)^2}$.

12. **(4 points)** Compute the total derivative of $f(x, y) = \begin{pmatrix} e^{xy} \\ x + y \end{pmatrix}$.
   - A. The total derivative is $\begin{pmatrix} ye^{xy} & xe^{xy} \\ 1 & 1 \end{pmatrix}$.
   - B. The total derivative is $\begin{pmatrix} e^{xy} & e^{xy} \\ x & y \end{pmatrix}$.
   - C. The total derivative is not defined for this function.
   - D. The total derivative is $\begin{pmatrix} \frac{1}{y}e^{xy} & x\frac{1}{x}e^{xy} \\ 1 & 1 \end{pmatrix}$.
   - E. The total derivative is $\begin{pmatrix} e^{xy} & e^{xy} \\ 1 & 1 \end{pmatrix}$.
   - F. The total derivative is $((x + y)e^{xy}, 2)$.

13. **(6 points)** We consider the ODE $\frac{\mathrm{d}y}{\mathrm{d}t} = -y + t$, and the corresponding explicit finite difference scheme. Which of the following statements are true?

    A. The explicit Euler method yields the discretization $y_{n+1} = \frac{y_n + (\Delta t) t_{n+1}}{1+\Delta t}$.

    B. The explicit Euler method yields the discretization $y_{n+1} = \frac{y_n + (\Delta t) t_{n+1}}{\Delta t}$.

    C. The explicit Euler method yields the discretization $y_{n+1} = y_n - (\Delta t)(-y_n - t_n)$.

    D. The explicit Euler method yields the discretization $y_{n+1} = y_n + (\Delta t)(-y_n + t_n)$.

    E. Explicit schemes are more computationally expensive (than implicit schemes).

    F. Explicit schemes usually have stability conditions.

14. **(4 points)** Use the method of Lagrange multipliers to find the extreme values of the function $f(x, y) = x^2 + y^2 + y$ subject to the constraint $x^2 + y^2 = 1$.

    A. There are no maximum or minimum values.

    B. The maximum value is 4, the minimum value is 1.

    C. The maximum value is 2, the minimum value is -2.

    D. There is no maximum value, and the minimum value is -1.

    E. The maximum value is 2, and there is no minimum value.

    F. The maximum value is 2, the minimum value is 0.

15. **(6 points)** We consider 2π-periodic complex-valued functions f, and their Fourier series $F_f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx}$. Which of the following statements are true?

    A. The Fourier coefficients $\hat{f}_k$ are defined as $\hat{f}_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} \, \mathrm{d}x$.

    B. Bessel's inequality $\sum_{k=-\infty}^{\infty} |\hat{f}_k|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \, \mathrm{d}x$ only holds for some functions f but not for all.

    C. Bessel's inequality $\sum_{k=-\infty}^{\infty} |\hat{f}_k|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \, \mathrm{d}x$ holds.

    D. The Fourier series $F_f(x)$ converges to $f(x)$ in the mean-square sense.

    E. The Fourier coefficients $\hat{f}_k$ are defined as $\hat{f}_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-kx^2} \, \mathrm{d}x$.

    F. The Fourier series $F_f(x)$ converges pointwise to $f(x)$.

16. **(15 points)**

We consider the function

$$f(x) = \frac{(x+1)^2}{(x-2)^2}.$$

(a) (1 point) What is the domain of the function?

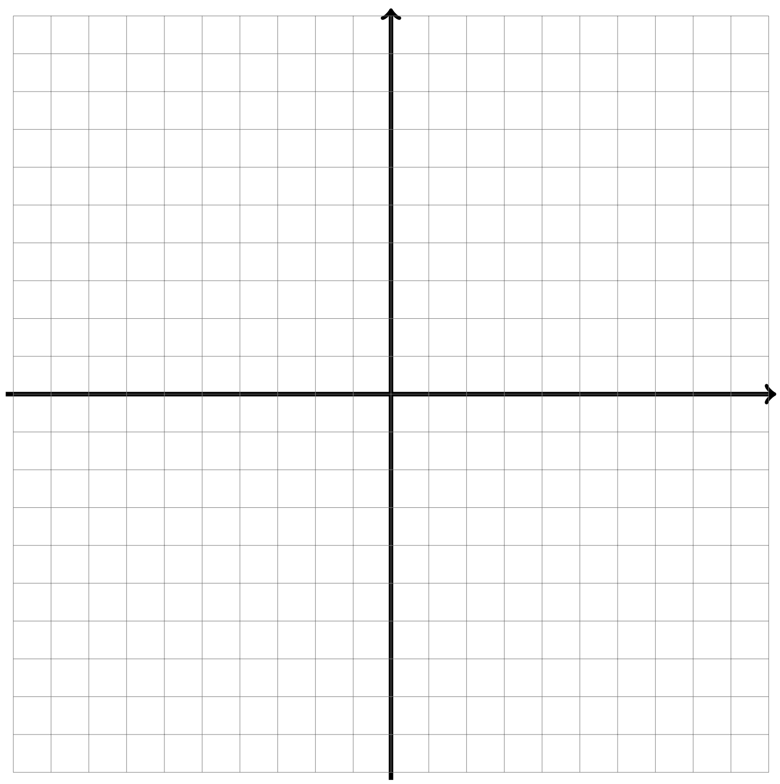
(b) (2 points) What are the intercepts with the x-axis and with the y-axis?

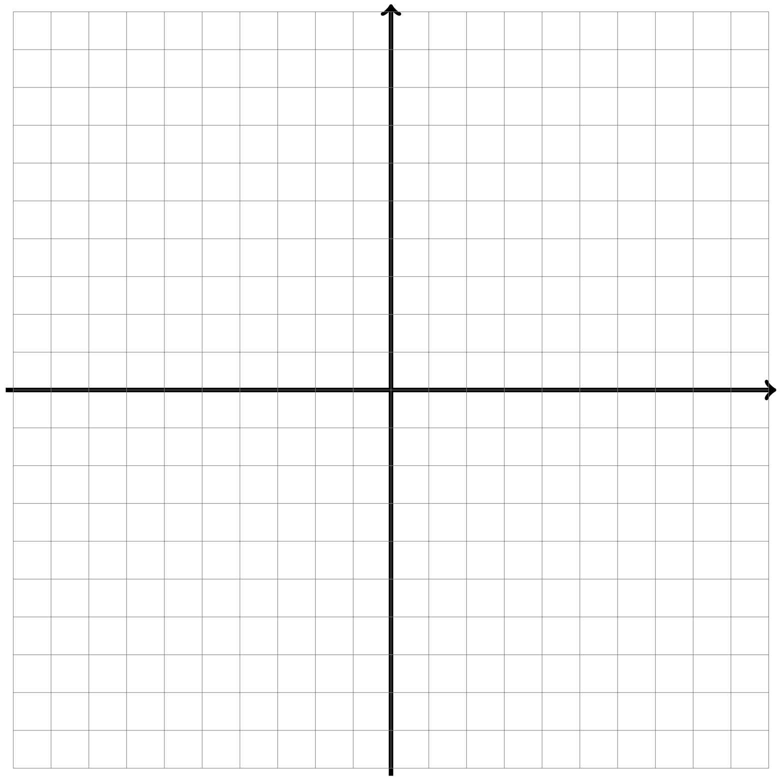
(c) (1 point) What are the horizontal asymptotes?

(d) (1 point) What are the vertical asymptotes?

(e) (5 points) Compute and analyze the first derivative. In which intervals is the function increasing or decreasing, what are the local minima or maxima?

(f) (5 points) Sketch the function. Your drawing needs to include all the qualitative features of the graph discussed in the questions above.





17. **(15 points)**

We consider the function

$$f(x, y) = 2 + x^3 + y^3 - 6xy.$$

(a) (7 points) Find all critical points of the function.

(b) (8 points) For each critical point, use the second derivative test in terms of the Hessian matrix to determine whether they are maxima, minima, saddle points, or something else.

18. **(15 points)**

We consider the linear homogeneous ODE

$$y'' + 2y' - 3y = 0$$

for a function $y(x)$.

(a) (5 points) Find the general solution to the ODE.

(b) (2 points) Provide the solution for the initial condition $y(0) = 1$ and $y'(0) = 2$.

(c) (1 point) What is the behavior of the solution as $x \to \infty$?

(d) (5 points) Find one particular solution to the linear inhomogeneous ODE

$$y'' + 2y' - 3y = e^{-2x}.$$

(e) (2 points) Provide the general solution (i.e., involving two constants) to the inhomogeneous ODE.