Constructor University | May 21, 2025
Spring 2025

Elements of Calculus
Practice Exam

Instructions:

- For the multiple choice questions, it is sufficient to mark the final answer(s) only. (No solution steps necessary.)
- For the longer exercises 16, 17, and 18, you need to show your work, i.e., carefully write down the steps of your solution. You will receive points not just based on your final answer, but on the correct steps in your solution.
- No tools or other resources are allowed for this exam. In particular, no notes and no calculators.

1. **(4 points)** Let $f(x) = \frac{\cos x}{x^2}$. Which of the following statements are true?
   A. The range of f is the interval $[-1, 1]$.
   B. The domain of f is $\{x \in \mathbb{R} : x \neq 0\}$.
   C. $f(2\pi) = \frac{1}{4\pi^2}$.
   D. f is injective.
   E. f has a horizontal asymptote at $y = 0$.
   F. f is bijective.

2. **(4 points)** What is the radius of convergence ρ for the power series $\sum_{k=1}^{\infty} \frac{3^k}{2k+1} x^k$?
   A. $\rho = \frac{1}{3}$.
   B. $\rho = 0$.
   C. $\rho = \frac{1}{2}$.
   D. $\rho = 2$.
   E. $\rho = \infty$.
   F. $\rho = 1$.

3. **(4 points)** Evaluate the limit

$$\lim_{x \to 0} \left(x \ln x + e^{2x} x^4 \right).$$

   A. 1.
   B. e.
   C. $+\infty$.
   D. $-\infty$.
   E. 2.
   F. 0.

4. **(6 points)** Consider the function

$$f(x) = x \sin\left(\frac{1}{x}\right) \quad \text{for } x \neq 0,$$

and $f(0) := 0$. Which of the following is true?

A. $|f(x)| \leq |x|$.
B. $f\left(\frac{1}{\pi}\right) < 0$.
C. f is continuous at 0.
D. $f\left(\frac{1}{\pi}\right) > 0$.
E. f is not continuous at 0.
F. $f\left(\frac{1}{\pi}\right) = 0$.

5. **(4 points)** Calculate the derivative $f'(x)$ of the function

$$f(x) = \frac{e^x}{x}.$$

A. $f'(x) = \frac{xe^x}{(x-1)^2}$.
B. $f'(x) = \frac{(x-1)e^x}{x^2}$.
C. $f'(x) = \frac{(x-1)e^x}{x}$.
D. $f'(x) = \frac{e^x}{x^2}$.
E. $f'(x) = \frac{(x-2)e^x}{x^2}$.
F. $f'(x) = -\frac{e^x}{x^2}$.

6. **(4 points)** Calculate the derivative $f'(x)$ of the function

$$f(x) = \cos(x^2).$$

A. $f'(x) = 2x\cos(x^2)$.
B. $f'(x) = -\sin(2x)$.
C. $f'(x) = -2x\sin(x^2)$.
D. $f'(x) = 2x - \sin(x^2)$.
E. $f'(x) = \cos(2x)$.
F. $f'(x) = -\frac{1}{3}\sin(x^3)$.

7. **(4 points)** Which of the following is the Taylor series of $f(x) = \sinh(x) := \frac{e^x - e^{-x}}{2}$ around $x_0 = 0$?

   A. $\sum_{k=0}^{\infty} x^k$.

   B. $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$.

   C. $\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$.

   D. $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$.

   E. $\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$.

   F. $\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

8. **(4 points)** Compute the integral $\int_1^e \frac{\ln(x)}{x} \, \mathrm{d}x$. (Hint: One can use integration by substitution.)

   A. $\frac{1}{2}$.

   B. $\frac{e}{2}$.

   C. 0.

   D. e.

   E. 2.

   F. 1.

9. **(4 points)** Compute the integral $\int_0^{\pi} x \sin(x) \, \mathrm{d}x$. (Hint: One can use integration by parts.)

   A. 1.

   B. 0.

   C. $\frac{\pi}{2}$.

   D. 2.

   E. π.

   F. $\frac{1}{4}$.

10. **(4 points)** Compute the integral $\int_{-1}^{1} \frac{1}{x^2}\,\mathrm{d}x$. (Hint: This is an improper integral.)
    A. The integral does not exist.
    B. 2.
    C. 1.
    D. 0.
    E. -1.
    F. -2.

11. **(4 points)** Solve the ODE $\frac{\mathrm{d}y}{\mathrm{d}t} = -yt$ with initial condition $y(0) = 2$.
    A. $y(t) = 2e^{-t^2}$.
    B. $y(t) = 2e^{-\frac{t^2}{2}}$.
    C. $y(t) = 3e^{-t} - 1$.
    D. $y(t) = 2e^{-t}$.
    E. $y(t) = 2\ln t + e$.
    F. $y(t) = e^{-t}$.

12. **(4 points)** Compute the total derivative of $f(x, y) = \begin{pmatrix} x^3 + 2y \\ x^2 y \end{pmatrix}$.
    A. The total derivative is $\begin{pmatrix} 3x^2 & 2 \\ 2xy & x^2 \end{pmatrix}$.
    B. The total derivative is $3x^2 + 2 + 2xy$.
    C. The total derivative is not defined for this function.
    D. The total derivative is $\begin{pmatrix} 1 & 2x \\ 2x & 2y \end{pmatrix}$.
    E. The total derivative is $\begin{pmatrix} 3x^2 \\ 2xy \end{pmatrix} + \begin{pmatrix} 2 \\ x^2 \end{pmatrix}$.
    F. The total derivative is $(3x^2 + 2, 2xy + x^2)$.

13. **(6 points)** We consider the ODE $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y,t)$ for some given function f, and the corresponding explicit and implicit finite difference schemes. Which of the following statements are true?

    A. The explicit Euler method yields the discretization $y_{n+1} - y_n = \Delta t f(y_{n+1}, t_{n+1})$.

    B. Explicit schemes usually have stability conditions.

    C. Implicit schemes are usually unconditionally stable.

    D. The explicit Euler method yields the discretization $y_{n+1} - y_n = \Delta t f(y_n, t_n)$.

    E. Explicit schemes are more computationally expensive.

    F. Implicit schemes are only defined for exponential decay.

14. **(4 points)** Use the method of Lagrange multipliers to find the extreme values of the function $f(x,y) = x + y$ subject to the constraint $x^2 + y^2 = 1$.

    A. There are no maximum or minimum values.

    B. The maximum value is $\sqrt{2}$, the minimum value is $-\sqrt{2}$.

    C. The maximum value is 2, the minimum value is -2.

    D. There is no maximum value, and the minimum value is -1.

    E. The maximum value is $\sqrt{2}$, and there is no minimum value.

    F. The maximum value is 1, the minimum value is -1.

15. **(4 points)** Which of the following is a solution to the one-dimensional wave equation

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}.$$

    A. $u(x,t) = (x - ct)^2 + e^{x+ct}$.

    B. $u(x,t) = \sin(cx)\sin(t)$.

    C. $u(x,t) = \cos(cx)\cos(t)$.

    D. $u(x,t) = \cos(x^2 - c^2t^2)$.

    E. $u(x,t) = x^3 - c^2t^2$.

    F. $u(x,t) = Ae^x \sin(t + c)$ for some constant A.

16. **(15 points)**

We consider the function
$$f(x) = (x+1)^2 e^{-x}.$$

(a) (1 point) What is the domain of the function?

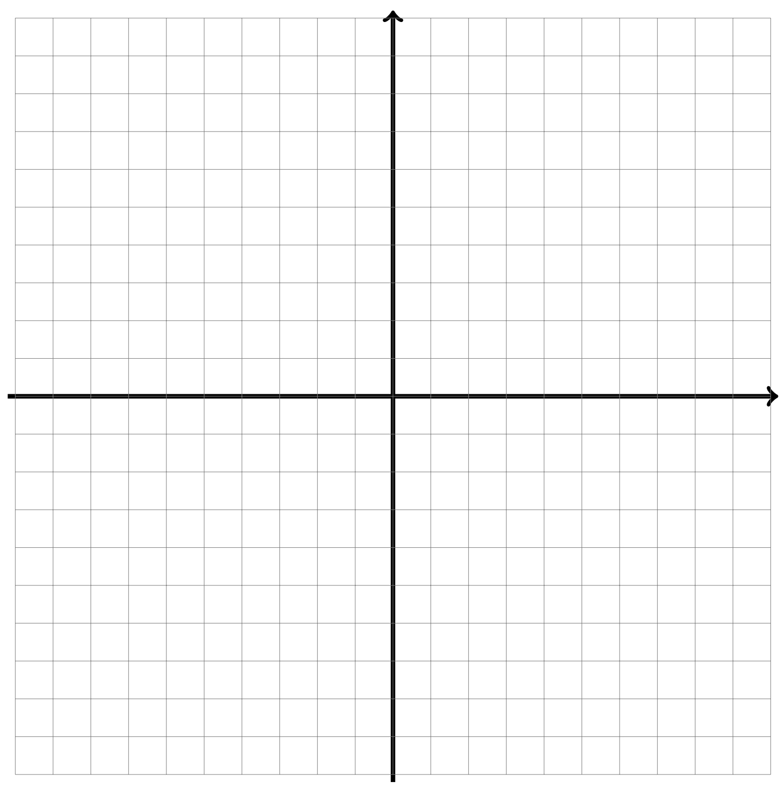
(b) (2 points) What are the intercepts with the x-axis and with the y-axis?

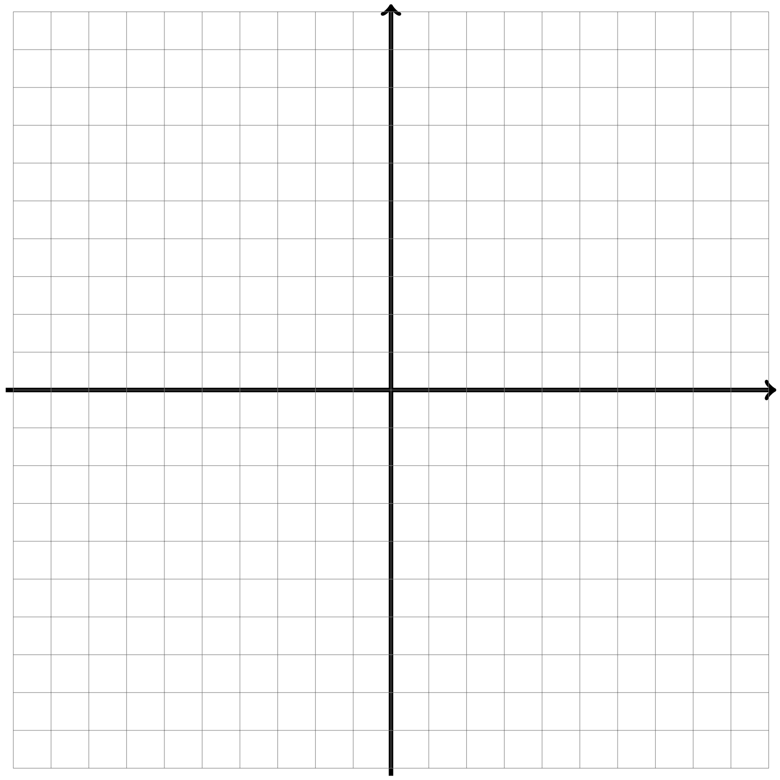
(c) (1 point) What are the horizontal asymptotes?

(d) (1 point) What are the vertical asymptotes?

(e) (5 points) Compute and analyze the first derivative. In which intervals is the function increasing or decreasing, what are the local minima or maxima?

(f) (5 points) Sketch the function. Your drawing needs to include all the qualitative features of the graph discussed in the questions above.





17. **(15 points)**

We consider the function

$$f(x, y) = xy(1 - x - y).$$

(a) (7 points) Find all critical points of the function.

(b) (8 points) Select two critical points, and use the second derivative test in terms of the Hessian matrix to determine whether they are maxima, minima, saddle points, or something else.

18. **(15 points)**

We consider the linear homogeneous ODE

$$y'' - 4y' + 3y = 0$$

for a function $y(x)$.

(a) (5 points) Find the general solution to the ODE.

(b) (2 points) Provide the solution for the initial condition $y(0) = -1$ and $y'(0) = 2$.

(c) (1 point) What is the behavior of the solution as $x \to \infty$?

(d) (5 points) Find one particular solution to the linear inhomogeneous ODE

$$y'' - 4y' + 3y = e^{-x}.$$

(e) (2 points) Provide the general solution (i.e., involving two constants) to the inhomogeneous ODE.