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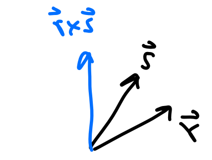
Lecture notes from Fall 2026

Review: Levi-Civita and cross product:

In $n=3$: cross product $\vec{r}\times\vec{s} = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} r_j s_k \underbrace{\hat{e}_i}$

Euclidean basis vector in direction i:

$\hat{e}_i := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ i-th coordinate



(right-hand rule)

↳ oriented area: • $|\vec{r}\times\vec{s}| = |\vec{r}|\,|\vec{s}|\sin\varphi$, φ = angle between $\vec{r}$ and $\vec{s}$

• direction orthogonal/normal to the area

$\left((\vec{r}\times\vec{s})\cdot(\vec{r}-\vec{s}) = 0\right.$, check$)$

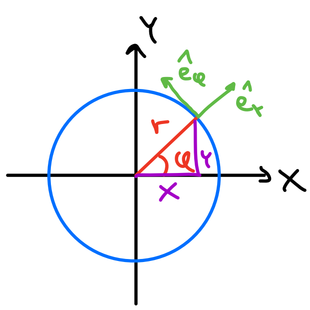
↳ oriented volume: $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$

↳ note: $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$ (HW)

Coordinate systems:

· Cartesian: $x\in\mathbb{R}, y\in\mathbb{R}, z\in\mathbb{R}$, volume element $dV = dx\,dy\,dz$

· polar coordinates: $r\in\mathbb{R}_0^+ := [0,\infty)$, $\varphi\in[0,2\pi)$, volume (area) element $dA = r\,dr\,d\varphi$



$x = r\cos\varphi$
$y = r\sin\varphi$, i.e., $\begin{pmatrix} x \\ y \end{pmatrix} = \vec{\Phi}(r,\varphi) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$

inverse: $\begin{pmatrix} r \\ \varphi \end{pmatrix} = \vec{\Phi}^{-1}(x,y) = \begin{pmatrix} \sqrt{x^2+y^2} \\ \arctan\frac{y}{x} \end{pmatrix}$

Recall: • Jacobian J_ϕ is $J_\phi = \begin{pmatrix} \frac{\partial \Phi_1}{\partial r} & \frac{\partial \Phi_1}{\partial \varphi} \\ \frac{\partial \Phi_2}{\partial r} & \frac{\partial \Phi_2}{\partial \varphi} \end{pmatrix}$, i.e., $(J_\phi)_{ij} = \partial_j \Phi_i$

partial derivative w.r.t. j-th argument

$$\Rightarrow J_\phi = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$$

• basis vectors: $e_r = \frac{\partial}{\partial r}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\partial}{\partial r}\vec{\Phi}(r,\varphi) = \begin{pmatrix} \frac{\partial \Phi_1}{\partial r} \\ \frac{\partial \Phi_2}{\partial r} \end{pmatrix} = J_\phi\, \hat{e}_x$

$$e_\varphi = \frac{\partial}{\partial \varphi}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi_1}{\partial \varphi} \\ \frac{\partial \Phi_2}{\partial \varphi} \end{pmatrix} = J_\phi\, \hat{e}_y$$

$$\Rightarrow e_r = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

$$e_\varphi = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \end{pmatrix}$$

$\Rightarrow$ normalized basis vectors: $\hat{e}_r = \frac{e_r}{|e_r|} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$

$|e_r| =$ length of $e_r = \sqrt{e_r e_r}$

$$\hat{e}_\varphi = \frac{e_\varphi}{|e_\varphi|} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} \qquad \left(|e_\varphi| = \sqrt{r^2\sin^2\varphi + r^2\cos^2\varphi} = r \right)$$

• volume (area) element $dA = \underbrace{|\det J_\phi|}\, dr\, d\varphi = r\, dr\, d\varphi$

$$= \left|\det\begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}\right|$$

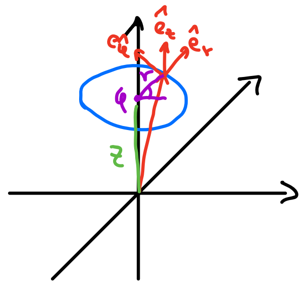
$$= \left| r\cos^2\varphi + r\sin^2\varphi \right| = r$$

Note: • $\hat{e}_r, \hat{e}_\varphi$ change with position

• bad at origin 0: it maps to ∞ many coordinates $(0,\varphi)$.

• Cylindrical (3d): $r \in \mathbb{R}_0^+, z \in \mathbb{R}, \varphi \in [0, 2\pi)$

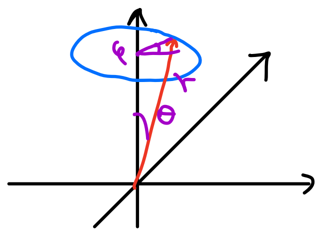
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix}$$



volume element: $dV = r\,dr\,d\varphi\,dz$ (check)

• Spherical (3d): $r \in \mathbb{R}_0^+, \varphi \in [0, 2\pi), \theta \in [0, \pi]$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\varphi\sin\theta \\ r\sin\varphi\sin\theta \\ r\cos\theta \end{pmatrix}$$



volume element: $dV = r\,dr\,d\varphi\,\sin\theta\,d\theta$

A few remarks about manifolds.

Manifold = $\underbrace{\text{space}}_{\text{a Hausdorff and 2nd countable topological space}}$ that is locally Euclidean ($=\mathbb{R}^n$)

More precisely: • chart $\phi_i : U_i \to \mathbb{R}^n$ continuous and bijective (= homeomorphism)
$\qquad U_i \subset \mathcal{M}$

• atlas: set of charts s.t. $\bigcup_i U_i = \mathcal{M}$ and $\phi_{ij} := \phi_i \circ \phi_j^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ smooth
(∞ often diff.able)

Interesting manifolds in classical physics:

- Euclidean space $\mathbb{R}^2, \mathbb{R}^3$
- 2d sphere S^2: need at least two charts to cover S^2!

  E.g., use stereographic projection
- Minkowski spacetime $\mathbb{R}^{3,1}$: $\mathbb{R}^4$ with metric $g = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$

- Configuration space $C = \mathbb{R}^{3n}$: space of all n particle positions in $\mathbb{R}^3$

  More general: space of all "degrees of freedom" (e.g., n particles on sphere: $C = (S^2)^n$)
- Phase space T^*C. For n particles in Euclidean space $= \mathbb{R}^{6n}$ = space of all positions and momenta.

  Later: Has a very important natural structure: symplectic form
- Symmetry groups: one can identify continuous symmetries with manifolds