Analytical Mechanics — Session 3

Prof. Sören Petrat, Constructor University

Lecture notes from Fall 2026

---

2. Kinematics

"How things move."

position $\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = x(t)\hat{e}_x + y(t)\hat{e}_y + z(t)\hat{e}_z$

velocity $\vec{v}(t) = \dot{\vec{r}}(t) := \frac{d}{dt}\vec{r}(t)$

acceleration $\vec{a}(t) = \dot{\vec{v}}(t)$

Simple example: constant acceleration $\vec{a}$

$\Rightarrow \quad \ddot{\vec{r}}(t) = \vec{a}$

$\Rightarrow \quad \vec{v}(t) = \dot{\vec{r}}(t) = \vec{a}\,t + \vec{v}_0$, with $\vec{v}_0 = \vec{v}(0)$

$\Rightarrow \quad \vec{r}(t) = \frac{1}{2}\vec{a}\,t^2 + \vec{v}_0 t + \vec{r}_0$, with $\vec{r}_0 = \vec{r}(0)$

3. Dynamics

"Why things move."

3.1 Newton's laws:

1. First Law ("inertia"): There exist inertial reference frames (IRF).

An IRF is def. by: Objects in constant motion remain in constant motion, and objects at rest remain at rest, all until influenced by an external force.

2. Second Law ("force"): $\vec{F} = \frac{d\vec{p}}{dt}$, with $\vec{p} = m\vec{v}$ the momentum

↳ If $m = \text{const}$, then this reduces to: $\vec{F} = m\vec{a}$

↳ These are two differential equations for $\vec{r}(t)$ and $\vec{p}(t)$ ($m = \text{const.}$ here):

- $\frac{d\vec{r}(t)}{dt} = \frac{\vec{p}(t)}{m}$
- $\frac{d\vec{p}(t)}{dt} = \vec{F}(t)$

Given $\vec{F}(t)$ (often not so simple in practice, can also depend on $\vec{r}(t)$ and $\vec{v}(t)$) and initial conditions $\vec{r}(0) = \vec{r}_0$, $\vec{p}(0) = \vec{p}_0$, we need to solve for $\vec{r}(t)$ and $\vec{p}(t)$.

3. Third Law ("action-reaction"): When body 1 applies a force $\vec{F}_{12}$ on body 2, body 2 applies an equal force $\vec{F}_{21}$ on body 1 in opposite direction:

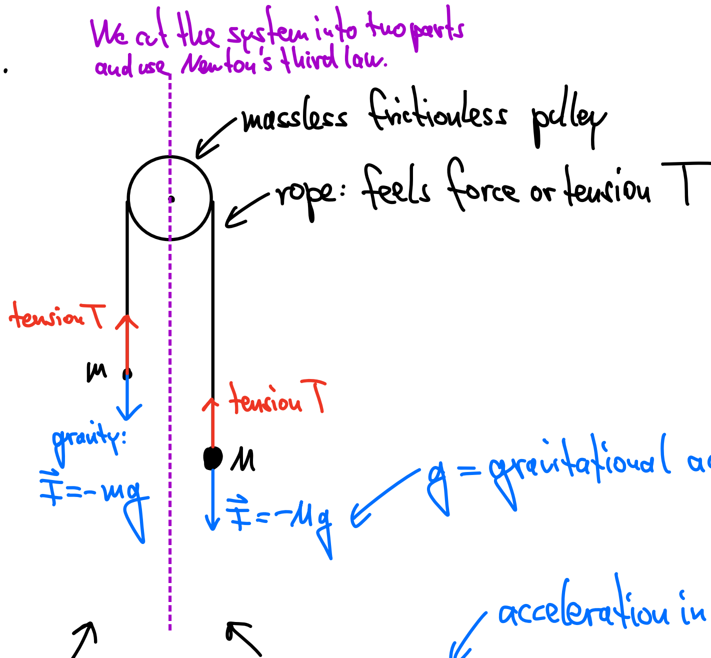
$$\vec{F}_{12} = -\vec{F}_{21}.$$

3.2 Free body diagrams

Free body diagram: minimalistic sketch of components of a system and involved forces.

Examples:

(1) Pulley.



g = gravitational acceleration near surface of earth, $g \approx 9.81 \frac{m}{s^2}$

acceleration in opposite direction

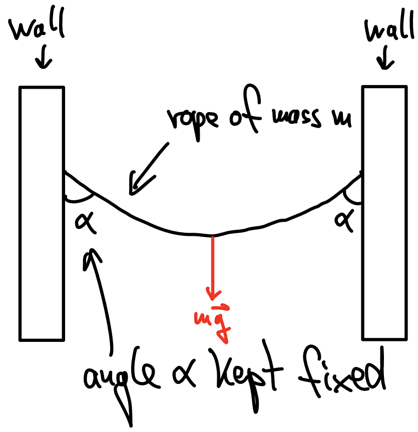
$$T - mg = ma \quad , \quad T - Mg = -Ma$$

$$a = \frac{T - mg}{m} \implies T - Mg = -M\left(\frac{T-mg}{m}\right) = -\frac{M}{m} T + Mg$$

Solve for T: $$\underbrace{\left(1 + \frac{M}{m}\right)}_{= \frac{m+M}{m}} T = 2gM \implies T = 2g \underbrace{\left(\frac{mM}{m+M}\right)}_{\text{reduced mass}} = g \underbrace{\frac{2}{\frac{1}{M} + \frac{1}{m}}}_{\text{harmonic mean}}$$

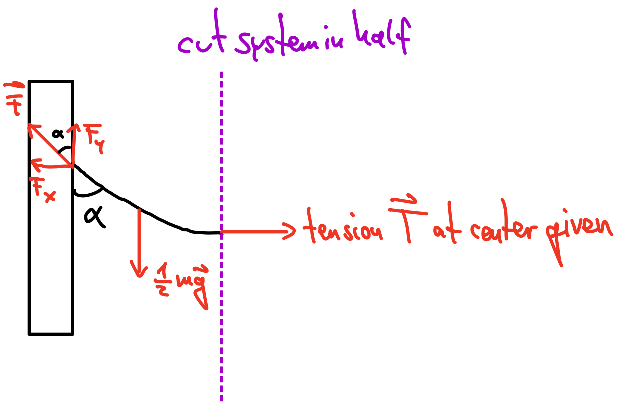
$$\implies \text{acceleration: } a = \frac{T}{m} - g = 2g \frac{M}{m+M} - g = g\left(\frac{2M}{m+M} - 1\right) = g\left(\frac{M-m}{M+m}\right)$$

(2) Rope.



Goal: Find α in terms of T

Configuration is static, so acceleration $\vec{a}=0$.

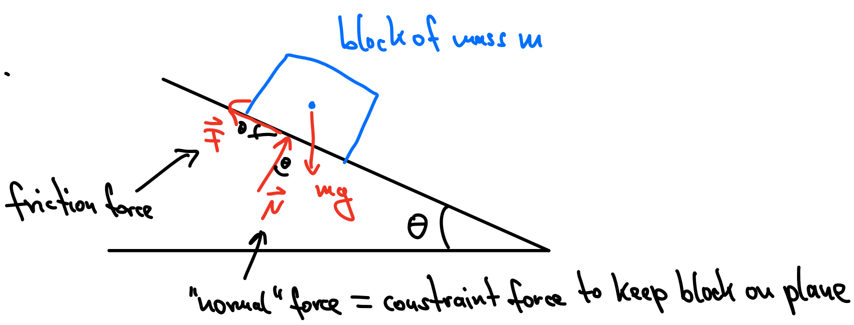


$$\text{x-direction: } T - |\vec{F}|\sin\alpha = \frac{1}{2} m a_x \overset{\text{equilibrium, } \vec{a}=0}{=} 0$$

$$\text{y-direction: } -\frac{1}{2}mg + |\vec{F}|\cos\alpha = \frac{1}{2} m a_y = 0$$

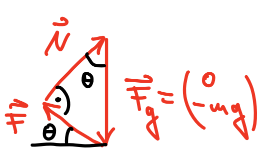
$$\Rightarrow \frac{|\vec{F}|\sin\alpha}{|\vec{F}|\cos\alpha} = \tan\alpha = \frac{T}{\frac{1}{2}mg} \quad \Rightarrow \quad \alpha = \arctan\frac{2T}{mg}$$

(3) Incline.



$$\vec{N} + \vec{F} + \begin{pmatrix} 0 \\ -mg \end{pmatrix} = m\vec{a} \overset{!}{=} 0$$

to compute friction force $\vec{F}$ s.t. no acceleration



$$\Rightarrow \quad |\vec{N}| = mg\cos\theta, \quad |\vec{F}| = mg\sin\theta = |\vec{N}|\tan\theta$$

↳ Static friction: $0 \leq |\vec{F}| \leq \mu_s |\vec{N}|$ as necessary to prevent slipping

$\Rightarrow$ if $\tan\theta \leq \mu_s$, the block stays at rest

↳ Dynamic friction: $|\vec{F}| = \mu_d |\vec{N}|$ ↗

$\tan\theta < \mu_d$: deceleration
$\tan\theta > \mu_d$: acceleration
$\tan\theta = \mu_d$: constant velocity slipping