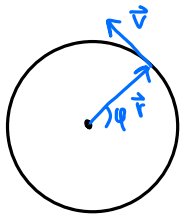
Prof. Sören Petrat, Constructor University

Lecture notes from Fall 2026

3.4 Angular momentum and torque

Angular velocity $\vec{\omega} := \frac{1}{|\vec{r}|^2}\, \vec{r} \times \vec{v}$

Example: Suppose a particle moves around the origin:



In spherical coordinates: $\vec{r} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ 0 \end{pmatrix}$, $\vec{v} = \frac{d\vec{r}}{dt} \overset{\frac{dr}{dt}=0}{=} \begin{pmatrix} -r\dot{\varphi}\sin\varphi \\ r\dot{\varphi}\cos\varphi \\ 0 \end{pmatrix}$

$$\Rightarrow \vec{\omega} = \frac{1}{r^2} \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ 0 \end{pmatrix} \times \begin{pmatrix} -r\dot{\varphi}\sin\varphi \\ r\dot{\varphi}\cos\varphi \\ 0 \end{pmatrix} = \dot{\varphi} \begin{pmatrix} 0 \\ 0 \\ \cos^2\varphi + \sin^2\varphi \end{pmatrix} = \dot{\varphi}\, \hat{e}_z$$

Angular momentum: $\vec{\ell} := \vec{r} \times \vec{p}$

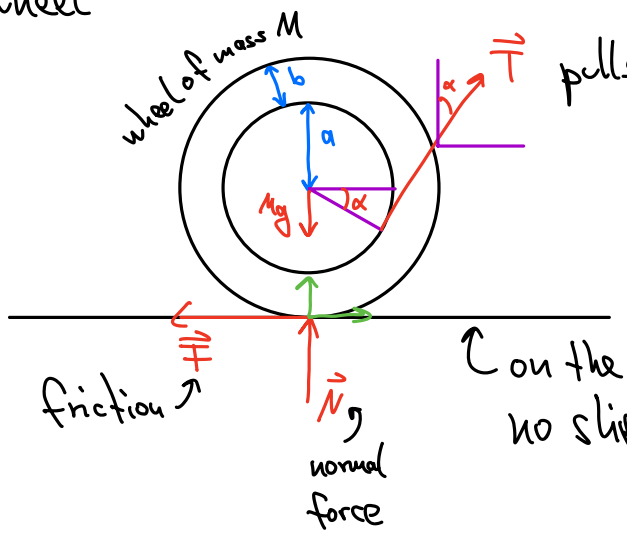
Torque: $\vec{\tau} = \vec{r} \times \vec{F}$

With these def.s: $\frac{d\vec{\ell}}{dt} = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} = \underbrace{\vec{v} \times (m\vec{v})}_{=0} + \vec{r} \times \vec{F} = \vec{\tau}$, analogous to Newton II.

Note: • $\vec{\ell} = \vec{r} \times \vec{p} = m\, \vec{r} \times \vec{v} = \underbrace{m|\vec{r}|^2}_{=: I = \text{moment of inertia}} \vec{\omega}$

• $\vec{\omega}, \vec{\ell}, \vec{\tau}$ all def. w.r.t. a reference point

Example: wheel



$\vec{T}$ pulls wheel at angle α

on the surface only rolling is allowed, no slipping (very high friction coefficient)

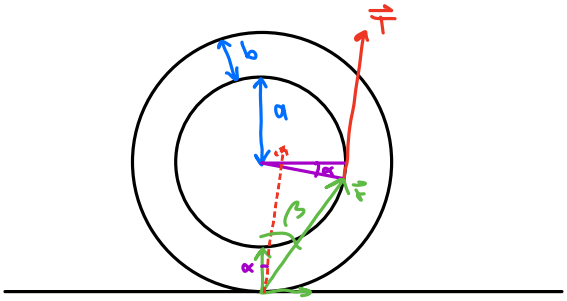
Question: At what angle α does the wheel get lifted instead of roll?

No roling $\Rightarrow$ torque $\vec{\tau} = 0$

Choose $\vec{0}$ = point where wheel touches ground.

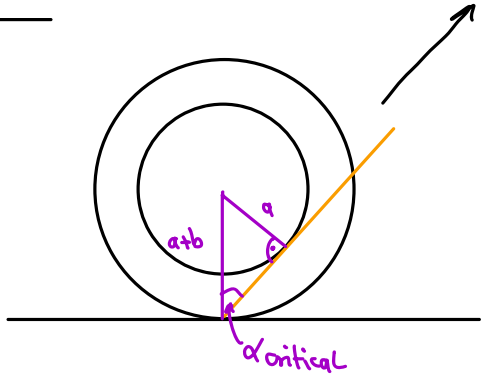
$\Rightarrow$ $\vec{F}$ and $\vec{N}$ are applied at $\vec{r} = 0$, so $\vec{\tau}_{\vec{F}} = 0 = \vec{\tau}_{\vec{N}}$. Also $\vec{\tau}_{mg} = 0$ since $\vec{r}(\text{center of mass}) \parallel \begin{pmatrix} 0 \\ -Mg \end{pmatrix} = \vec{F}_{grav}$.

$\Rightarrow$ $\vec{\tau}$ comes only from $\vec{T}$



$|\vec{\tau}| = |\vec{r}|\,|\vec{T}| \sin(|\beta - \alpha|) = 0$ if $\alpha = \beta = \alpha_{critical}$

We have $\sin \alpha_{critical} = \frac{a}{a+b}$.



↳ $\alpha = \alpha_{critical}$: no rolling

↳ $\alpha < \alpha_{critical}$: lift-off (and rolls counterclockwise)

↳ $\alpha > \alpha_{critical}$: wheel rolls forward (to the right)