

Linear Algebra

Homework 5

Due on March 11, 2026, before the tutorial.

Problem 1 [4 points]

Given $f \in \mathcal{L}(V, W)$ for finite-dimensional vector spaces V, W and its matrix in chosen bases of V and W , what is the matrix of the dual map in the dual bases?

Problem 2 [2 points]

Let V_1, V_2, W be subspaces of a vector space V . Either prove or give a counterexample to the following assertions.

- (a) If $V_1 + W = V_2 + W$ then $V_1 = V_2$.
- (b) If $V = V_1 \oplus W$ and $V = V_2 \oplus W$ then $V_1 = V_2$.

Problem 3 [6 points]

Let $V_1, \dots, V_n \subset V$ be subspaces of the vector space V with $\sum_{i=1}^n V_i = V$. Prove that the following three statements are equivalent:

(a)

$$V = \bigoplus_{i=1}^n V_i \quad (\text{i.e., } V \text{ is actually a } \textit{direct} \text{ sum of the } V_i),$$

(b)

$$V_j \cap \left(\sum_{\substack{i=1 \\ i \neq j}}^n V_i \right) = \{0\} \quad \forall j = 1, \dots, n,$$

(c)

$$\sum_{i=1}^n \dim V_i = \dim V.$$

Problem 4 [4 points]

Let V be a vector space. We call $p \in \mathcal{L}(V)$ a projector if $p^2 = p$. Now let $p_1, \dots, p_n \in \mathcal{L}(V)$ be projectors with $\sum_{i=1}^n p_i = \text{id}$ and $p_i p_j = 0$ for all $i \neq j$. Prove that then

$$V = \bigoplus_{i=1}^n \text{im}(p_i).$$

Problem 5 [4 points]

Let V be a vector space and let $p \in \mathcal{L}(V)$ be a projector. Prove that $V = \ker(p) \oplus \operatorname{im}(p)$. Show also that any projector can be represented in an appropriate basis by the matrix

$$A_p = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix},$$

where E_r is the $r \times r$ unit matrix with $r = \dim \operatorname{im}(p)$, and the 0's stand for $(n - r) \times (n - r)$ matrices with 0 entries ($n = \dim V$).