Constructor University
Spring 2026

March 16, 2026

Linear Algebra

Homework 7

Due on March 25, 2026, before the tutorial.

Problem 1 [16 points]

Let M be a subspace of L.

(a) Prove that $\dim M + \dim M^\perp = \dim L$.

(b) Prove that $(M^\perp)^\perp$ as a subspace of L^{**} is canonically isomorphic to M (under the canonical isomorphism that identifies elements of L^{**} with elements of L).

(c) In class we discussed a big diagram claiming ten canonical isomorphisms. Prove all of these. One point for each. (*Hint: Each one follows quite easily from applying the lemmas discussed in class and part (a) and (b).*)

Problem 2 [2 points]

For $f \in \mathcal{L}(L, M)$, prove that $f(x) = y$ has a solution if and only if y is orthogonal to the kernel of the dual map $f^* : M^* \to L^*$.

Problem 3 [2 points]

Prove that the column rank of a matrix (the maximal number of linearly independent column vectors) equals its row rank (the maximal number of linearly independent row vectors). Proceed by using that if a map $f : L \to M$ is represented in some basis by a matrix A, then the dual map in the dual basis is represented by the transposed matrix A^t.