

# Linear Algebra

## Homework 9

Due on April 22, 2026, before the tutorial.

### Problem 1 [4 points]

For the following two matrices

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix},$$

find the eigenvalues, their multiplicities, the corresponding eigenspaces or generalized eigenspaces and state whether they are diagonalizable or not. If a matrix is not diagonalizable, write down the Jordan normal form (only the matrix is enough).

### Problem 2 [3 points]

Solve the linear system of equations

$$y'(t) = J_n(\lambda)y(t),$$

where  $y(t) \in \mathbb{R}^n$ , and  $J_n(\lambda)$  is a Jordan block with  $\lambda \in \mathbb{R}$  on the diagonal. *Hint: One could do this by making a guess for the solution and then prove it by induction.*

### Problem 3 [2 points]

Consider the linear system of equations

$$y'(t) = Ay(t),$$

where  $y(t) \in \mathbb{R}^n$  and  $A$  is any complex  $n \times n$  matrix. Describe how to solve this equation by bringing  $A$  into Jordan normal form.

### Problem 4 [6 points]

Let  $L$  be a finite dimensional complex vector space and  $g \in \mathcal{L}(L)$ . An operator  $f \in \mathcal{L}(L)$  such that  $f^2 = g$  is called a square root of  $g$ .

- Suppose  $g$  is nilpotent. Prove that then  $1 + g$  has a square root. *Hint: Make a guess of what the square root should be from the Taylor series expansion for  $\sqrt{1+x}$ .*
- Now suppose that  $g$  is invertible. Prove that then  $g$  has a square root. *Hint: Start with the abstract Jordan decomposition and consider  $g|_{G(\lambda_i)}$ .*

**Problem 5 [5 points]**

Let  $L$  be a real or complex vector space.

- (a) Recall or look up the definition of a norm  $\|\cdot\|$  on  $L$ . Let us next consider any norm for linear operators  $f : L \rightarrow L$ . Let  $P(t) = \sum_{i=0}^{\infty} c_i t^i$  have non-zero radius of convergence. We set  $P(f) = \sum_{i=0}^{\infty} c_i f^i$  if the series  $\sum_{i=0}^{\infty} c_i \|f^i\|$  converges (in  $\mathbb{R}$ ).
- (b) Let  $f \in \mathcal{L}(L)$  be such that  $\|f\| < 1$ . Compute  $\sum_{i=0}^{\infty} f^i$  and thus prove that  $1 - f$  is invertible.
- (c) Let  $f, g \in \mathcal{L}(L)$ . We define  $e^f = \sum_{i=0}^{\infty} \frac{f^i}{i!}$ . Prove that when  $fg = gf$  (i.e.,  $f$  and  $g$  commute), then  $e^{f+g} = e^f e^g$ . *Hint: Cauchy product.*