Constructor University
Spring 2026

May 4, 2026

# Linear Algebra

## Homework 12

Due on May 13, 2026, before the tutorial.

**Problem 1 [3 points]**

Consider again the matrix

$$\begin{pmatrix} 3 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 1 \end{pmatrix}$$

from last week's homework sheet as the Gram matrix of a symmetric bilinear form on a real vector space in some basis. Write down the corresponding quadratic form and apply the following reduction scheme to reduce it as a sum of squares:

- Write (for $a_{11} \neq 0$):

$$\begin{aligned} q(x_1, \ldots, x_n) = \sum_{i,j} a_{ij} x_i x_j &= a_{11} \Big( \underbrace{x_1 + \frac{a_{12}}{a_{11}} x_2 + \ldots \frac{a_{1n}}{a_{11}} x_n}_{=:y_1} \Big)^2 + q'(\underbrace{x_2}_{=:y_2}, \ldots, \underbrace{x_n}_{=:y_n}) \\ &= a_{11} y_1^2 + q'(y_2, \ldots, y_n). \end{aligned}$$

- Repeat for $y_2, \ldots, y_n$.

**Problem 2 [3 points]**

Let $L$ be the real vector space of polynomials of degree 3 or less with bilinear form

$$g(f_1, f_2) = \int_{-1}^{1} f_1(x) f_2(x)\, dx.$$

What is the Gram matrix of $g$ in the basis $\{1, x, x^2, x^3\}$? Find an orthonormal basis of $L$. (With the Gram-Schmidt orthogonalization these are the first few Legendre polynomials.)

**Problem 3 [6 points]**

We consider the bilinear form

$$g(f_1, f_2) = \int_{-\infty}^{\infty} f_1(x) f_2(x) G(x)\, dx$$

with $G(x) = e^{-x^2}$. Then the result of the Gram-Schmidt orthogonalization applied to $\{1, x, x^2, \ldots\}$ are the Hermite polynomials $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$.

(a) Prove their normalization $g(H_n, H_m) = 2^n n! \sqrt{\pi} \delta_{nm}$.

(b) The Hermite functions are defined as

$$\psi_n(x) = (-1)^n (2^n n! \sqrt{\pi})^{-1/2} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2}.$$

Show that they are orthonormal with respect to the bilinear form

$$\widetilde{g}(f_1, f_2) = \int_{-\infty}^{\infty} f_1(x) f_2(x)\, dx.$$

(c) Show that the Hermite functions satisfy the differential equation for the quantum harmonic oscillator (the time-independent Schrödinger equation), i.e.,

$$\left( \frac{d^2}{dx^2} + 2n + 1 - x^2 \right) \psi_n(x) = 0.$$

**Problem 4 [2 points]**

Let $L$ be a vector space over $\mathbb{R}$. Prove the following two statements which illustrate the connection between norms and scalar products.

(a) If a norm $\|\cdot\|$ on $L$ is defined via a scalar product ($\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$) then it satisfies the parallelogram identity

$$\|\ell_1 + \ell_2\|^2 + \|\ell_1 - \ell_2\|^2 = 2\|\ell_1\|^2 + 2\|\ell_2\|^2.$$

(b) **[Bonus, 3 points]** A bit more challenging is the converse statement. Suppose for some norm $\|\cdot\|$ on $L$ the parallelogram identity holds. Then there is a scalar product $\langle \ell, \ell \rangle$ such that $\|\cdot\| = \sqrt{\langle \ell, \ell \rangle}$.

**Problem 5 [6 points]**

Let $L$ be a vector space over $\mathbb{R}$. We show that not every metric is defined from a norm, and not every norm from a scalar product.

(a) Let $L = \mathbb{R}^n$ with $n \geq 2$. Prove that $\|\ell\| := \max\{|x_i| : i = 1, \ldots, n\}$ is a norm, but that there is no scalar product with $\|\ell\| = \sqrt{\langle \ell, \ell \rangle}$.

(b) Let $L$ be the vector space of real continuous functions. For any $k \in \mathbb{N}$, define $\|f\|_k := \max\{|f(x)| : x \in [-k, k]\}$. Prove that

$$d(f, g) := \sum_{k=0}^{\infty} 2^{-k} \frac{\|f - g\|_k}{1 + \|f - g\|_k}$$

is a metric on $L$, but there is no norm $\|\cdot\|$ such that $\|f - g\| = d(f, g)$.